APPLICATION OF FINITE ELEMENT METHOD FOR CONTINUUM MECHANICS PROBLEMS

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BY
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CERTIFICATE

This is to certify that the work entitled 'APPLICATION OF FINITE ELEMENT METHOD FOR CONTINUUM MECHANICS PROBLEMS' by Arup Chattopadhyay has been carried out under my supervision and has not been submitted elsewhere for a degree.

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NOMENCLATURES

All quantities in this study are defined with respect to the coordinate system at undeformed state.

CHAPTER 1:

A_{ij}, A_{ijk}	Rivlin-Ericksen tensors for mono and dipolar fields.
E_{ij}, E_{ijk}	Strains
e	Internal energy density
F^i, F^{ij}	Body forces
G^{ij}, g^{ij}	Metric tensors for undeformed and deformed state
h	Heat flux
K^{ij}	Conductivity tensor
n_k	Components of unit outward normal
P^i, P^{ij}	Surface loads
Q^i	Heat flux vector
q	Distributed energy sources inside the body per unit mass.
T^{ij}, T^{ijk}	Stresses
${}_dT^{ij}$, ${}_dT^{ijk}$	Dissipative parts of the stresses
$T^{(ij)}$ etc.	Functionals for stresses
$T^{(ij)}$, $T^{\|ij\|}$	Symmetric and antisymmetric parts of T^{ij},
t	Time
U_i, V_i	Displacements and velocities
$V_{i\|j}$	Velocity gradient where bar denotes covariant derivative

X^i, $\overline{X}$, Z^i	Curvilinear coordinates, Position vector and Cartesian coordinates at reference state
x^i, $\overline{x}$	Coordinates and position vector for deformed state
α	Indeces
δ^k_j	Kronecker delta
η	Entropy
θ	Temperature
ν_α	Thermodynamic affinities
ρ_o, ρ	Mass densities at undeformed and deformed state
τ^α	Thermodynamic stresses
ψ	Helmholtz free energy

CHAPTER 2

$C1_{\alpha\beta}$, $C2_{\alpha\alpha'}$, $C3_{\beta\beta'}$, $C4_{\gamma\gamma'}$	Matrices in governing Eqs. for finite element discretization
$D_{i\alpha}$, S^{ij}_β, Φ_γ	Interpolation functions for displacements, stresses and temperature
$L1_\alpha$, $L2_\beta$, $L3_\beta$, $L4_\gamma$, $L5_\gamma$, $L6_\gamma$	Loading vectors
$U_{\alpha N}$, $T_{\beta N}$, $\theta_{\gamma N}$	Nodal values of displacements, stresses and temperature .
$\overline{U}^*$, $\overline{L}^*$ etc.	$\frac{\partial \overline{U}}{\partial \lambda}$, $\frac{\partial \overline{L}}{\partial \lambda}$ etc.
λ	Parameter used in parametric differentiation scheme

CHAPTER 3

A_m	Area of integration for m-th element
a, b	Dimensions of the rectangular element
$C^{(ji)(kl)}$, $\tilde{C}^{(ji)(kl)}$, $\underset{\sim}{C}^{(ji)}$ C^{ij}_{θ}, $\tilde{C}^{ij}_{\theta}$, C^{θ}_{θ}	Material coefficients defined Material coefficients defined in Eqs. (3.3-8), (3.3-11) and (3.3-12)
$I^1_{\beta\alpha}$, $I^2_{\beta\underset{\sim}{\alpha}}$, $\tilde{I}^1_{\beta\alpha}$, $\underset{\sim}{I}_{\beta\gamma}$, J_p, K_p, $\tilde{K}_p$, $R_{\gamma\gamma'}$, $S_{\gamma\alpha}$, $Y_{\gamma\gamma'}$, $\underset{\sim}{Y}_{\gamma\gamma'}$, ${Y^1}_{\gamma\alpha'}$	Different matrices
X^1, X^2	Cartesian coordinates
Y, Y^i	A typical quantity and its nodal values

CHAPTER 4

A, $\underset{\sim}{l}$	Domain of integration over the surface and boundary of an element
A_{mn}, B_{mn}, C_{mn}	Nodal unknown vectors for displacements and temperature for n-th harmonic.
$\tilde{C}_{ij}$, Q_{ij}, C_i	Constant material coefficients defined in Eq. (4.3-3)
D_m	Interpolation function
G^{kn}_{im}, $\underset{\sim}{G}^{k}_{ip}$, I^k_{ip}, $\underset{\sim}{\tilde{X}}^{kn}_{ip}$, $\tilde{\ }^{kn}_{im}$	Expressions used to define incremental stresses in Eq. (4.4-5)
$H^1_n(\theta)$, $H^2_n(\theta)$	Harmonic functions
K_{ij}, $\tilde{K}_{ij}$, $\underset{\sim}{K}_{ij}$	Matrices used in the governing equation (4.3-18)
{L}	Overall load vector
r, θ, z	cylindrical coordinates

T	Temperature
T_i	Single subscripted stresses
T_i^o, $\bar{T}_i$	Linear and nonlinear parts of stresses
$\bar{T}_i^*$, $\bar{\epsilon}_i^*$ etc.	$\frac{\partial \bar{T}_i}{\partial}$, $\frac{\partial \epsilon_i}{\partial}$ etc.
X_{im}, $\tilde{X}_{im}^k$, $X_{\sim im}^k$	Functions used to define linear stresses
$\{Z\}$	Overall nodal unknown vectors
γ_i	Rivlin-Ericksen tensors
ϵ_i	Strains
ϵ_i^o, $\bar{\epsilon}_i$	Linear and nonlinear parts of strains
λ	Parameter used for Parametric differentiation scheme

CHAPTER 5

A, L	Cross sectional area and length of the bar
A_m	Area of the m-th element
a	A typical dimension in the cross-section
$\{b\}$	Nodal unknowns, in general.
$\|C\|$, $\|H\|$, $\|S\|$, $\|\bar{S}\|$	Different matrices
G	Shear modulus
m, p	typical node and element
s	Contour parameter of the boundary
T	Torque
U_{co}	Complementary energy
X^1, X^2, X^3	Cartesian coordinates
γ, τ	Resultant strain and stress

$\gamma_1, \gamma_2, \tau_1, \tau_2$	Strains, stresses
γ_o, τ_o	Nondimensionalizing factors for strains and stresses
ϵ	Strain vector
$\langle \zeta \rangle$	Interpolation function for complementary energy approach
θ	Twist per unit length
π	Complementary potential energy
$\{\sigma\}$	Stress vector
$\hat{\tau}_i$	Functionals for stresses
$\emptyset$	Stress function
ψ	Warping function

CHAPTER 6

$a_{21}, a_{22}, a_{31}, a_{32}$	Dimensions of an element
w, β_1, β_2	Transverse deflection and otations
$M_1, M_2, M_{12}, V_1, V_2$	Moments and shears
A, A_m	Domain of integration for the entire plate and for the m-th element respectively
A_j, A_1, A_2	Pseudo load term arising out of the error terms
$\bar{B}$	8 x 8 Differential operator matrix
$\bar{D}, D$	Exact and approximate state vector
$D^m, K^m, L^m, t, T, \emptyset, \Phi$	Vectors or matrices defined for finite element derivation.
$\hat{D}, \tilde{D}, \hat{F}, \tilde{F}$	Force and displacement vectors on boundary
E	Young's modulus

E^m	Unknown loading vector due to discretization error defined in equation
K^m_{ij}	Partitioned matrices of K^m
$\bar{L}$	Load vector
s	Path of line integration
s_p	Length of path from p-th to p+1st node on s.
S_d, S_σ	Boundary where displacements or stresses are specified
$x^{\alpha i}_j$	$a_{\alpha j} - a_{ij}$
ζ , H	Typical quantities defined in relevant connections
ϵ^m	Exact error vector for m-th element
ϵ^m_i	Discretization error in i-th node for m-th element
λ	A scalar factor
ν	Poisson's ratio
Ω	Boolean transformation

CHAPTER 7

A	Curvilinear area of the element
CF_1-CF_7, H^+,H^-	Expressions defined in Eq. (7.7-2)
E	Young's modulus
H, H_i	Typical quantity and its nodal values
I^i_1-I^i_{12}, I^{ij}_{13}-I^{ij}_{46}	Integrals defined in Table 7-1
L^i_1 - L^i_7	Loading integrals defined in Table 7-1.

M_x, M_y, M_{xy}	
N_x, N_y, N_{xy}	Moment and stress resultants
V_x, V_y etc.	
M_{xk}, N_{xk}, V_{xk}, N_{xyk} etc.	Nodal values of moment and stress resultants
p_x, p_y	Membrane loads
q^+, q^-	Normal loads
R_x, R_y	Principal radii of curvatures of middle surface
$\overset{*}{T}_i$	Interpolating functions
$u_x, u_y, \beta_x, \beta_y, w$	Deformation components
$u_{xk}, u_{yk}, \beta_{xk}, \beta_{yk}, w_k$	Nodal values of deformation components
x, y, z	Curvilinear coordinates
α_x, α_y	Lame's parameters for curvilinear surface
ν	Poisson's ratio

SYNOPSIS

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The principal objective of this study is to develop a sequence so that it will be possible to analyse matter as available in nature, having certain geometric properties and environmental stipulations under suitable approximation that are not over restrictive. With this idea in mind, Chapter 1 has been devoted for the formulation of the general equations in continuum mechanics. In this chapter, Section 1 deals with the reference frame, characterising the variant state of a body, while Section 2 gives the principle of material frame indifference, which a physical law should always satisfy. In Section 3 the type of forces and stresses used in this sequel, has been defined through the concept of power generation having conjugates as velocities and velocity gradients. In subsequent sections, the governing equations for a continuum has been derived through the principles of thermodynamics and material frame indifference for dipolar stress field which has been summarized in the last section.

In Chapter 2, the equations based on finite element approximation and parametric differentiation have been developed. The first few sections of this chapter deals with the historical review of finite element technique, requirements in the formulation and the derived equations based on general interpolation formula. These equations, in general, will be highly complex and nonlinear with respect to chosen field variables. To solve this set of equations, parametric differentiation technique has been adopted. This will reduce the set of equations into another linear set of first order differential equations with variable coefficients in terms of the incremental field variables with respect to a certain chosen parameter. These equations together with a standard quadrature technique can be employed to solve the problem.

In Chapter 3, the basic equations derived in Chapter 2, have been simplified for two dimensional plane problem choosing rectangular finite element model. A few problems concerning elastic,viscoelastic and thermal cases have been solved for numerical example. Chapter 4 has been devoted to the simplification in the case of axisymmetric solids of revolution, discretised by ring shaped triangular finite element model having only linear distribution of displacements, expanded in fourier series in the circumferential directions. Examples have been provided by the solutions of a few harmonic

loadings for isotropic and anisotropic linear material with linear geometry. Nonlinear cases have been shown for axi-symmetric loading and also for viscoelastic material. Chapter 5 deals with the pure torsion problem with linear and higher order (HCT) stress function distribution over a triangular element. Numerical examples have been given for different cross sections and nonlinear materials.

In Chapter 6 and 7, mixed triangular models have been developed for plate (in cartesian coordinates) and shell (in curvilinear coordinates) assuming linear distribution for all the displacements and stresses. In these two chapters, Reissner's variational approach, or more precisely, simplified equations of Naghdi's formulations (with shear correction) have been employed to get the equations for the finite element. Moreover, error analysis and convergence criteria for the plate model has also been studied by application of Taylor series expansion and 'classical order of accuracy' method. Finally, concluding remarks and comments on future research have been given in Chapter 8.

GENERAL INTRODUCTION

Keeping pace with the progress of science, the technologists want to set up and solve their problems more and more precisely and accurately. For pracise formulation and for the solution of problems mathematics become an indispensable tool. But it is also of immense importance to remember that the set of equations representing a physical system describes only a mathematical model which is used to approximate the physical reality to a certain degree. It is usually possible to state the physical processes in terms of relations between some variables, which generally appear in such equations as physical parameters and are to be determined experimentally. These experiments become meaningful only when they can be performed with sufficient accuracy and iduplicability and can be analysed with reasonable precision for the state of stress and deformation in the test specimen under suitable approximations which are not over restrictive. The study concerning the behaviour of such specimen which is essentially a form of matter available in nature, has its origin in the concept of thermodynamics[1].

The science of thermodynamics[2,3] may be broadly said to deal with (1) energy and its transformations and (2) the concept of equilibrium and its evolution with particular reference to

systems involving thermal effects. This is merely a convenient way of classifying all applications of thermodynamics and indicates how broad a field it serves. 'Pure' thermodynamics is based on two fundamental laws. These laws were developed and extensively tested in the latter half of nineteenth century and may be said to rest on a very broad foundation of experience. A third law has been developed during the present century; but although of a fundamental character and based on sound experimental and theoretical grounds, it has nothing like the scope of application of the other two laws. It is sufficient for the present purpose to note that, whereas the first two laws led to the definition of new and useful functions, the third law, in essence, merely limits the value of one of these functions (entropy). Since it is outside the present scope of study, henceforth no reference will be made about the third law of thermodynamics.

All naturally occurring changes or processes are 'irreversible' in the sense that they tend to proceed in a certain direction and never, when left to themselves, to reverse, or go in the opposite direction. Like a clock they tend to 'run down' and cannot 'rewind' themselves. These are matters of common experience. Nevertheless, the second law of thermodynamics is nothing more or less than a generalized statement of such common experience. However, many of the

changes in nature are reversible, at least partially. The important point is that things can be done only at the expense of some other system, which itself runs down. Work must be done to reverse these changes, and the work can be obtained only by allowing a spontaneous change to occur somewhere else. To state this another way, it is possible to undo the result of a given spontaneous change only by allowing another such change and useful work can be secured only when the tendency of a spontaneous change exists.

The tendency to change can be recognized as being due to certain 'driving forces'. The change may not appear to take place because of some resistance that may slow up the change as to make it imperceptable. If the existence of the driving force is recognized, then it is possible to predict the direction of the change.

The end point of any spontaneous process is a state of equilibrium in which forces are balanced and there is no further tendency to change. It is important to make a distinction between a true, or stable, equilibrium with its balanced forces and an apparent, or false, equilibrium in which the tendency to change still exists but, owing to a high resistance, the rate of change is so small that to all intents and purposes no change is taking place, i.e., cessation of motion does not necessarily indicate a true

equilibrium. Thus, it is seen that there is difference between 'true equilibrium' in thermodynamic point of view and 'equilibrium' from mechanical concept.

In mechanics, there are two other laws characterizing the mechanical state of a system. They are known as Cauchy's laws of motion. One of them states the conservation of momentum and the other furnishes the conservation of moment of momentum.

Unfortunately, the thermodynamic principles, in themselves, are not sufficient to prove the Cauchy's laws of motion. Although from a special thermodynamic consideration, the first law can be derived, but it is not possible to derive for the second law. 'The equations of mechanics describe a wider range of phenomena than do the principles of thermodynamics. No one will contest the balance of mass, momentum and energy, but the existence of a caloric equation of state is an assumption of a more special kind'[1]. In fact the principal objection regarding thermodynamic laws lies in the concept and existence of entropy in nonequilibrium state.

As long as a system is in equilibrium, the classical thermodynamics (not statistical) are logical and self contained. But the situation is essentially different for the systems which represent a more wider irreversible nonequilibrium process. Here, the question arises whether to give up

entirely the concept of entropy and adopt an 'entropy free' thermodynamics or to extend the definition of entropy to states far from equilibrium[4,5]. However, even without intruding deep into the fundamental validity of thermodynamics, it may be stated that some other hypothesis is necessary to derive the Cauchy's laws from the principles of thermodynamics. The hypothesis adopted here is the objectivity principle of physical laws.

The objectivity principle plays an important role in the theory of relativity. Here this principle is applied on space-time continuum. In classical mechanics, the velocities of different systems are considered to be very very small when compared with the velocity of light. In this situation, time measured in different intertial frames is invariant. It will seen that when this simplified objectivity principle is applied on the laws of thermodynamics, the Cauchy's laws and thermal equation of state will be naturally separated so as to ensure the objectivity of the laws. The subjects upto the derivation of the general laws of motion and heat equation will be dealt in Chapter 1.

Once these general equations are obtained, it is a matter of technique to get a solution out of them for a definite problem. Theoretically it is possible but the amount of complexities encountered for the exact solution are

virtually prohibitive. Even the classical approximate processes such as asymptotic expansions and weighted residual techniques can be applied to relatively simpler problems. Due to the advent of high speed computers in last few decades, the approximate techniques such Ritz, finite difference, weighted residue, invariant imbedding etc. again have come to limelight. But each one of them have got their inherent shortcomings. Perhaps, the finite element method may be considered as one of the most powerful and versatile discretization presently available among the approximate techniques.

Numerical methods, as such, are very useful because, when properly applied, they can be used to solve within engineering accuracy for the unknown field variables of extremely complex systems. Historically, difficult engineering problems have been approached by simplifying the system to one that could be solved using either simple formulas or the closed form solution of the governing equations for the much modified system. Of course, analytical complications may arise in many problems. The difficulties, generally encountered, are the following: First the overall character of the system has been altered, and second, in the zones where steep gradients of the field variables exist, the fidelity of the solution is questionable. Hence, an accurate modelling of these situations is very desirable. As the functions

which the system performs becomes more critical, the need for a less approximate method of analysis becomes more imperative. Therefore numerical analysis techniques such as finite element method have developed in an attempt to model the real system more closely.

Using the finite element method, the system is imagined as being divided into a number of finite regions, or elements. The behavior of the individual element is assumed approximately. Then the behavior of entire system is assembled from that assigned to the individual elements. Satisfaction of complicated geometry and boundary conditions do not pose a very great problem. Neither the finite difference nor lumped parameter methods possess this advantage to the same degree as the finite element method. Even the zone of high gradients can be more simply modeled using finite elements, though it always requires special attention. Thus, the finite element technique is selected over other discrete methods of analysis.

There are various alternative ways for obtaining the equation resulting from finite element discretization. By far the most popular method is based on variational principles with respect to minimization of potential or complementary energy or Reissner's energy formulation. However, any other variational approach of mathematical physics is equally applicable. In this sequel, Galerkin's technique has been

applied in the first few chapters where the approximate equations have been obtained from the general derivations of Chapter 1. A part of the torsion analysis which has been dealt in Chapter 5, the principle of minimum complementary energy has been utilised whereas the equations for plates and shells have been derived through Reissner's approach.

Finite element technique annihilates the space dependence and expresses the field equations in terms of the field variables at discrete points, called nodes. The resulting equations, in general, will be nonlinear and simultaneous which obviate the use of any direct method of solution.Hence, necessity arises for the adoptation of a systematic incremental approach. A parametric differentiation scheme has been chosen here as the method of stepwise solution of the equations. This is adopted because it appears to be quite logical and straight forward for application. In a nut-shell, the method consists of differentiation of the governing equations with respect to a parameter which is assumed to be the only independent variable of the solution space where a solution is represented by a point in this space. The external proportional loading in the case of time independent system or time in the case of time dependent system may be selected as this parameter. The resulting set of equations will be linear

and simultaneous with respect to the differentials and can be solved. Now it is only left to calculate the actual values of the field variables for a small increment of the parameter using any quadrature formula. This way, the solution will march out upto a required value of the parameter. The convergency of this process is somewhat dubious. But the question seems to be related to the order of the quadrature formula and the step length, both of which can be improved in the computer programming at the expense of execution time. The finite element and the associated parametric differentiation scheme have been presented in Chapter 2.

Different applications to structural problems have been shown in Chapters 3 to 7 with numerical examples and the general conclusion with comments on future research has been presented in Chapter 8. A note for the computer programs used for solving different problems has been given in Appendix A.

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CHAPTER 1

GOVERNING EQUATIONS

1.0 Introduction:

Matter is commonly available in nature in the form of materials. Analytical mechanics elude this fact, introducing concepts of mass point and rigid body which is independent of its constitution. Modern physics concerns with the elementary particles of matter. It evades the question how a specimen, made up of such particles, will behave in natural circumstances. The first approach deals with the study of rigid body, whereas, the second approach concerns with atomic physics. A theory aiming to bridge this gap and to furnish the matter as available in nature must describe their mechanical behaviour, deformability and represent the definite laws they obey under certain approximations which are not over restrictive.

The complete mathematical statement describing the state of a deformable medium consists of momentum and mass conservation, thermodynamic laws, certain geometric properties and environmental stipulations expressed as boundary and initial conditions on loads, displacements and/or heat flow. The variables commonly used to describe these changes of state are stress and displacement components, temperature,

entropy and heat flux. The variant state of a body must be specified through frames of reference, which has been discussed in Section 1. Moreover, the laws of nature should not be special for a frame (or an observer). This objectivity principle is dealt in Section 2. In Section 3, different types of forces relevant for subsequent analysis have been defined through the principle of power generation.

The whole domain of classical mechanics, which is the basic background of the present work, is based on the approximate Newtonian concept and the classical law of non-interconvertibility of mass and energy. But there exists various forms of energy. They are amenable to change their form during a change of state or so to say, during a process. Any process will be governed by thermodynamic laws (by definition). In actuality, as any other laws, they will put restrictions on the inter diffusion of various form of energy in matter, e.g., mechanical, heat, electrical magnetic etc. In this sequel, only the first two types will be considered.

The ultimate idea of continuum mechanics is to predict the behaviour of matter during a process, as closely as human knowledge permits. Since the present progress speaks in all certitude that matter is discrete and energy transformations for the elementary particles follow only

probabilistic certainty even for very slow and gross motion, it leads to a direct contradiction of the principle of classical thermodynamics and consequently of continuum mechanics. A happy reconciliation can be made if statistical thermodynamics is introduced. But the added complexity due to the introduction of statistical mechanics may not be commensurate with the accuracy achieved in a sense of solution of general practical problems. However, though the present motivation is in no way so ambitious, a procedure will be adopted so that more and more exactness can be incorporated to meet the ever increasing accuracy required by future technologists.

With this idea in mind, Section 4 is devoted for thermodynamic concepts and conservation laws. In Sections 5, 6 and 7, the balance of energy will be considered in details and the laws of motion will be derived from the principle of material frame indifference. Section 8 deals with the equation of thermal state, which will be deduced from the expression of entropy. The general physical laws, in themselves, do not suffice to determine the deformation of a body. Before a determinate problem can be formulated, it is usually necessary to specify the material of which the body is made. In continuum mechanics, such specification is stated by constitutive equations. A very brief discussion on constitutive equations will be made in Section 9. For convenience and future references, the important equations

which will be needed in subsequent analysis, will be summarised in Section 10.

1.1 Kinematic Preliminaries:

To study the motion of a continuum, it is necessary to establish the correlations between different phases of the motion which the continuum undergoes with respect to time and relative to a reference state. The description of such a reference state must be characterized through a set of time dependent coordinate frames.

Let B be a three dimensional manifold of material points within the material volume v and bounded by the surface s at initial time $t = t_o$. The position of a material point X in this region may be denoted by one of the coordinate frames, i.e., a rectangular cartesian coordinate Z^i, a curvilinear coordinate X^i or by a position vector $\bar{X}$, where

$$X^i = X^i(Z^1, Z^2, Z^3) = X^i(Z^j)$$

$$\bar{X} = \bar{X}(X^i) = \bar{X}(X^i(Z^j))$$

and the indeces having latin miniscules (i, j, k etc.) take the value of 1, 2 and 3 and refer to the reference coordinate system in the undeformed state.

A motion of the body B is the family of mappings with respect to time parameter t of B into Euclidian space B_t.

The image,

$$\bar{x} = \bar{x}\,(X^i, t) \tag{1.1-1}$$

of X is called the position vector of X at time t. In component form, Eq. (1.1-1) can be written as,

$$x^i = x^i\,(X^j, t) = x^i\,(Z^j, t) \tag{1.1-2}$$

having four independent variables t and X^i or Z^i, i = 1,2 and 3.

It is assumed that mappings (1.1-2) are single valued and have continuous partial derivatives with respect to their arguments to whatever order desired except possibly at isolated singularities like points, curves and surfaces which necessitate special attentions. Further each member of (1.1-2) has unique inverse of the other in a neighbourhood of a material point X. Hence position vectors $\bar{X}$ and $\bar{x}$ may also be expressed as function of x^i at any instant of time. But to save space and to avoid any imbroglio which may appear in subsequent derivations for finite elements, all quantities, in this sequel, will referred to undeformed state X^i as has been shown in Eqs. (1.1-1) and (1.1-2). Further axioms of indestructibility and impenetrability of matter will lead to the inequality,

$$\left| \frac{\partial x^i}{\partial X^j} \right| > 0 \tag{1.1-3}$$

Any deformation of continuum, treated in this variant, will be tacitly assumed to respect this inequality.

The covariant components of base vectors and metric tensor for the undeformed and deformed configurations may be defined as,

$$\bar{G}_i = \bar{G}_i\ (X^j) = \frac{\partial \bar{X}}{\partial X^i} \qquad (1.1\text{-}4)$$

$$\bar{g}_i = \bar{g}_i\ (X^j,\ t) = \frac{\partial \bar{x}}{\partial X^i} \qquad (1.1\text{-}5)$$

and
$$G_{ij} = \bar{G}_i \cdot \bar{G}_j \qquad (1.1\text{-}6)$$

$$g_{ij} = \bar{g}_i \cdot \bar{g}_j \qquad (1.1\text{-}7)$$

The contravariant components of the corresponding quantities can be obtained by solving the equations

$$G^{ij} \cdot G_{jk} = \delta^i{}_k;\ \bar{G}^i = G^{ij}\ \bar{G}_j \qquad (1.1\text{-}8)$$

and
$$g^{ij} \cdot g_{jk} = \delta^i{}_k;\ \bar{g}^i = g^{ij}\ \bar{g}_j \qquad (1.1\text{-}9)$$

where $\delta^i{}_k$ denotes Kronecker delta.

Instead of $\bar{x}$, the position vector of time t can also be expressed in terms of the deformation vector $\bar{U}$, such that

$$\bar{x} = \bar{X} + \bar{U} \qquad (1.1\text{-}10)$$

where
$$\bar{U} = \bar{U}\ (X^i,\ t)$$

and
$$\bar{U}\ (X^i,\ 0) = 0 \qquad (1.1\text{-}11)$$

i.e., time is reckoned from the undeformed state. The covariant and contravariant components of the deformation

vector are,

$$\bar{U} = U^i \bar{G}_i = U_i \bar{G}^i \qquad (1.1\text{-}12)$$

and the classical strain tensor can be defined as,

$$\begin{aligned} E_{ij} &= \frac{1}{2} (g_{ij} - G_{ij}) \\ &= \frac{1}{2} (U_{i|j} + U_{j|i} + U_{k|i} U^k{}_{|j}) \end{aligned} \qquad (1.1\text{-}13)$$

where, bar denotes covariant differentiation with respect to metric G_{ij}.

It is worth mentioning here, that strain plays the role of an intermediary to correlate through thermodynamics, the actions and their corresponding response within matter. It will be seen later that certain quantities are necessary as arguments of functions so as to satisfy certain basic postulates of physics. There are a host of quantities[1,20], which might be acceptable and, as such, they may be called strains, but their fundamental dependence on displacement gradients are not important except to satisfy the physical aspects for which it is meant and relation (1.1-13) is only one of them, but will be used here to conform to traditions.

1.2 Material Frame Indifference:

It has been seen that to specify any event in a system, it is necessary that the event has to be located with respect to a reference frame providing information for space as well as time. The specification of a reference frame is not unique and depends upon different observers. But the fundamental measurable quantities, such as distances, angles and time intervals are independent of the observer. Consequently, any change of frame must preserve them along with the temporal orders of all the events. The most general relation for such a change of frame may be expressed as,

$$\begin{aligned} \bar{x}^* (X^k, t^*) &= R(t)\, \bar{x}\, (X^k, t) + \bar{D}(t) \\ t^* &= t - a \end{aligned} \tag{1.2-1}$$

where $(\bar{x}^*, t^*)$ represents an event in one frame and $(\bar{x}, t)$ in another. $\bar{D}(t)$ is a constant vector representing a point, $R(t)$ is a proper orthogonal tensor and a is a real number. $D(t)$ and $R(t)$ may be assumed as smooth function of t. It is not difficult to visualize that the two frames are related by a rigid transformation combined with a time shift and differ only by an Euclidian displacement. In cartesian coordinate system, Eq. (1.2-1) may be expressed as,

$$\begin{aligned} x^{*i} (Z^k, t^*) &= R^i{}_j(t)\, x^j(Z^k, t) + D^i(t) \\ t^* &= t - a \end{aligned} \tag{1.2-2}$$

Two motions of a given medium related by the Eq. (1.2-1) or (1.2-2) are said to be equivalent.

For a change of frame, transformation laws for scalar, vector and tensor are given as follows:

(a) A scalar remains unchanged.

(b) A vector is transformed into,

$$\bar{v}^{*}(X^k, t^*) = R(t) \cdot \bar{v}(X^k, t)$$

or

$$\bar{V}^{*i}(Z^k, t^*) = R^i{}_j(t)\, \bar{v}^j(Z^k, t) \tag{1.2-3}$$

(c) A second order tensor is transformed into,

$$S^*(X^k, t^*) = R(t).S(X^k, t).R^T(t)$$

or

$$S^{*i}{}_j(Z^k, t^*) = R^i{}_m(t)\, S^m{}_l(Z^k, t)\, R_j{}^l(t) \tag{1.2-4}$$

Functions and fields whose values are scalars, vectors and tensors will be called objective or frame indifferent, if both dependent and independent variables transform according to the above laws.

Equations governing the motion of a continuum will be derived primarily from a postulate that a physical law will always satisfy the principle of material frame indifference, applied on the balance of energy principle. A short history together with a general discussion on the principle of material frame indifference has been given in references 1, 2 and 3.

1.3 Definition of Forces:

A material body under the influence of external and internal forces undergoes deformations. On a quantitative basis, there may be three types of forces[1] which can influence a medium.

A. Extrinsic forces:

These forces are due to external effects and regarded as acting upon the particles comprising the body. The body force due to uniform gravity is a relevant example.

B. Mutual forces:

These forces arise within the body and are regarded as acting upon pairs of particles having equal magnitude but opposite directions. Therefore, the resultant internal force is zero. Force between two charged particles is an example.

C. Contact forces:

These forces result due to action of one body on another through the boundary surface, lines or points. Stress field and surface loads belong to this category.

The quantitative definition of forces can be derived easily from the concept of work rate. However, it is to be noted that the definition of forces is not unique nor straight forward. The basic idea is that certain affinities (e.g. velocities V_i) being acted upon by corresponding fluxes (forces, P^i) produce power (i.e., rate of work $\dot{W} = P^i V_i$). Now,

conversely if P^i is a contravariant vector and V_i is an arbitrary covariant components of a set of velocity and if the scalar $P^i V_i$ is a rate of work corresponding to the velocity vector, then P^i may be defined as a force in the direction of X^i. Thus defined, force may be assumed as a function of position vector, velocity and time. In continuum mechanics, this idea may be easily extended to define forces of more general nature[4,5].

Surface Force:

Let an independent set of kinematic variables with respect to motion (1.1-2) be denoted by a tensor function $X_{ii_1 \ldots i_\beta}$ under coordinate transformation, where,

$$x_{ii_1 \ldots i_\beta} = x_{ii_1 \ldots i_\beta} (X^k, t) \;, \; \beta = 1,2,3.. \quad (1.3\text{-}1)$$

In addition, if tensors (1.3-1) respect the principle of material frame indifference, then they may be designed as simple 2^β pole displacement field or in general multipolar displacement field[4]. Now, consider a surface s whose unit normal at point X is n_i. If $P^{ii_1 \ldots i_\beta \,:\, j_1 j_2 \ldots j_\alpha}$ is a tensor of order $\alpha + \beta + 1$ and if for all arbitrary 2^β pole velocity gradients $\dot{x}_{ii_1 \ldots i_\beta | j_1 \ldots j_\alpha}$, the scalar

$$P^{ii_1 \ldots i_\beta \,:\, j_1 \ldots j_\alpha} \cdot \dot{x}_{ii_1 \ldots i_\beta | j_1 \ldots j_\alpha}$$

is a rate of work per unit area of s, then tensor

$P^{ii_1\ldots i_\beta:j_1\ldots j_\alpha}$ may be called a surface force $2^{\alpha+\beta}$ -pole of the $(\beta + 1)$ th kind per unit area of that surface[5]. When $\beta = 0$, $P^{i:j_1\ldots j_\alpha}$ denotes the surface force 2^α-pole of the first kind and when $\alpha = 0$, $P^{ii_1\ldots i_\beta}$ is called a simple surface force 2^β-pole of $(\beta + 1)$ th. kind. $\alpha = \beta = 0$ corresponds to the classical surface force P^i. Various other degenerated cases can be obtained depending on the order of the tensor $x_{ii_1\ldots i_\beta}$, order of gradients α and the special properties imparted to $x_{ii_1\ldots i_\beta}$ (for example $x_{ii_1} = -x_{i_1 i}$). Such special cases and their relations with the director theories has been discussed by Green and Naghdi[6]. In this study, only the dipolar ($\alpha = 1$) force of first kind ($\beta = 0$) will be considered. Hence, the total rate of work over the surface s by the monopolar surface force P^i and dipolar surface force P^{ij} corresponding to the velocity V_i and velocity gradient $V_{j|i}$ will be given by,

$$\int_s (P^i V_i + P^{ij}\, V_{j|i})\, ds, \qquad (1.3\text{-}2)$$

Stresses:

Again consider a surface s_i normal to X^I-axis in the initial configuration. The corresponding contravariant components of the stress (monopolar) and hyperstress (dipolar) of first kind may be respectively denoted by T^{ij} and T^{ijk}, if the corresponding rate of work done is given by,

$$T^{ij}\, V_j \text{ and } T^{ijk}\, V_{k|j} \qquad (1.3\text{-}3)$$

per unit area of s_i for any arbitrary velocity V_j and velocity gradient $V_{k|j}$. The first index i of the stress and hyperstress components indicates the surface on which they act.

Body forces:

If F^i and F^{ij} are contravariant tensors and V_i and $V_{j|i}$ are arbitrary set of velocity and velocity gradient and if the scalars F^iV_i and $F^{ij}V_{j|i}$ are rates of work per unit mass, then F^i and F^{ij} are respectively called body forces of monopole and dipole of first kind per unit mass. The total rate of work of the body forces distributed throughout the volume v is given by,

$$\int_v \rho_o \left(F^iV_i + F^{ij}V_{j|i}\right) dv \tag{1.3-4}$$

where ρ_o is the density of the mass at time $t = t_o$.

1.4 Thermodynamic Preliminaries:

The basic idea of continuum mechanics is to formulate theories which will reflect, during any change of state, the overall expositions of the microscopic behaviour of matter. Moreover, for obtaining the complete system of equations describing the motion of the medium, it is necessary to know the general laws governing the changing phases of any medium and the characteristics of the particular medium.The necessary informations should come from the general principles of thermodynamics.

A thermodynamic system may be defined[7] as a system in nature whose action on its surroundings proceeds by way of output or absorption of work and heat. This definition applies even to a nonequilibrium process for which the equilibrium state is only a particular case. It characterizes thermodynamics by operations, i.e., the exchange of work and heat between the systems and its surroundings and it relates these quantities with the reaction of the system, which are deformations, the temperature and their gradients. Now a system may include discretely the fundamental particles of matter or it may specify larger aggregates exposing only the average reactions of particles which is the basis for a mathematical model of a continuum.

Since this study is based on continuum mechanics approach having no chemical and electromagnetic effect on the medium, it will be assumed that any physically small fundamental element of the medium can be regarded as a thermodynamic system. For each of these small elements, the mechanical concept of internal states are defined and each successive state of evolution is characterized by a finite number of defining parameters. The exchange of work and heat will be not only over the surface of total system but also with the surroundings of any specified subsystem.

Any fundamental element at any stage of evolution will follow three basic laws[3]:

1. Conservation of mass:

The mathematical expression for conservation of mass may be expressed as

$$\rho\,(\bar{x},\ t) \ = \rho_0(\bar{X},\ t_0)/\left|\frac{\partial x^k}{\partial X^l}\right| \tag{1.4-1}$$

where $\rho(\bar{x},\ t)$ is the density of mass at time t for a point $\bar{x}$ which was at $\bar{X}$ at time t_o. It will be assumed here that any deformation from the reference state will always satisfy Eq. (1.4-1), or in other way, this relation may be accepted as a definition of density at time t for the point $\bar{x}$.

2. Balance of Energy:

Balance of energy may be expressed as,

$$\dot{K} + \dot{E} \ = \ P + \sum_{\alpha} F_{\alpha} \tag{1.4-2}$$

where K, E and P represent respectively the kinetic energy, internal energy and the power supplied by the external mechanical forces. F_α represents the mechanical equivalent of the αth kind of nonmechanical energy per unit time, e.g., heat energy. The dot denotes differentiation with respect to time. This law may be designated as the first law of thermodynamics.

In a given physical problem, P, F_α and K are generally well defined. The remaining quantity, i.e., the internal energy may be looked upon as the quantity which balances this equation. Since internal energy is also an additive

function, in terms of the internal energy density e per unit mass, it can be written as,

$$E = \int_V \rho_o e \, dv \tag{1.4-3}$$

The internal energy density e is a material frame indifferent scalar and depends only on thermodynamic variable.

3. Second law of Thermodynamics:

If M be the number of mechanical parameters ν_α, which influence the state of motion of a material element having specific internal energy e, then the second law of thermodynamics[1] asserts that for any given values of e and ν_α in a certain domain, there exists a quantity called entropy whose specific value η depends on e and ν_α, i.e.,

$$\eta = \eta \quad (\epsilon, \nu_\alpha, X^i) \tag{1.4-4}$$

The differential form of Eq. (1.4-4),

$$d\eta = \frac{1}{\theta} de + \sum_\alpha \frac{\partial \eta}{\partial \nu_\alpha} d\nu_\alpha \tag{1.4-5}$$

gives the concept of θ, called the absolute temperature, though the equation,

$$\frac{1}{\theta} = \frac{\partial \eta}{\partial e} \tag{1.4-6}$$

The two functions of state θ and η have the following properties:

I. θ is a positive number which is a function of the empirical thermometric temperature.

II. The entropy is an additive function and the sum of entropies of the parts is equal to the entropy of the whole system.

III. Change of entropy can be affected by change of heat energy and by production of entropy inside the system. If $d\overset{*}{Q}$ denotes the increase of heat energy at temperature θ, then the increase of entropy can be written as,

$$d\eta = \frac{d\overset{*}{Q}}{\theta} + d\eta_{ir} \tag{1.4-7}$$

where $d\eta_{ir}$ is the production of entropy inside the system.

IV. For any process in nature the production of entropy inside a system can not be negative, i.e.,

$$d\eta_{ir} \geq 0 \tag{1.4-8}$$

The equality will correspond to a reversible system while the inequality stands for irreversible process.

V. The entropy function must fulfil the Glandsdroff-Prigogine theorem[8],

$$\sum_{\alpha} \frac{D}{Dt} \left(\frac{\partial \eta}{\partial \nu_{\alpha}} \right) . \overset{\circ}{\nu}_{\alpha} \leq 0 \tag{1.4-9}$$

Now if it is assumed that all thermodynamic functional relations are differentiable as many times as needed and are invertible to yield one variable as a function of the others, then from (1.4-4), e can be written as,

$$e = e(\eta, \nu_{\alpha}, X^{i}) \tag{1.4-10}$$

The physical dimension of ν_α are that of mechanical units (say displacement gradients and higher gradients), but otherwise arbitrary. The temperature θ and the thermodynamic tension τ^α may be defined from (1.4-10) by,

$$\theta = \frac{\partial e}{\partial \eta} \quad \text{and} \quad \tau^\alpha = \frac{\partial e}{\partial \nu_\alpha} \tag{1.4-11}$$

It will be assumed that this temperature is the conventional temperature. The thermodynamic tensions will be shown later to be related with conventional stresses. The differential form of the Eq. (1.4-10) can be written as,

$$de = \theta d\eta + \sum_{\alpha=1}^{M} \tau^\alpha d\nu_\alpha \tag{1.4-12}$$

Since this equation is valid for any arbitrary differential increment of η and ν_α, as a special case, it will be valid for the actual motion of the particle X. Hence, the corresponding rate equation can be written as

$$\dot{e} = \theta\dot{\eta} + \sum_{\alpha=1}^{M} \tau^\alpha \dot{\nu}_\alpha \tag{1.4-13}$$

Sometimes, instead of internal energy density e, it is more convenient to use Helmholtz free energy per unit mass, ψ, which may be defined as,

$$\psi = e - \eta . \theta \tag{1.4-14}$$

so that,

$$\begin{aligned} \eta &= \eta(\theta, \nu_\alpha, X^i) = -\frac{\partial \psi}{\partial \theta} \\ \tau^\alpha &= \tau^\alpha(\theta, \nu_\alpha, X^i) = \frac{\partial \psi}{\partial \nu_\alpha} \end{aligned} \tag{1.4.15}$$

where,

$$\psi = \psi(\theta, \nu_\alpha, X^i)$$

The name free energy is used because, it is that portion of the energy which is available for doing work at constant temperature.

1.5 Energy Balance:

Consider an arbitrary material volume v of the continuum bounded by a surface s having unit outward normal $\bar{n}$ in the reference configuration. From Eq. (1.3-2) and (1.3-4), the rate of work done by the surface forces over the surface s and by the body forces within the volume v is given by,

$$P = \int_s (P^i V_i + P^{ij} V_{j|i})\, ds$$

$$+ \int_v (F^i V_i + F^{ij} V_{j|i})\, dv \qquad (1.5\text{-}1a)$$

Now if the dipolar kinetic forces, i.e., inertia forces due to gradient of velocities are neglected, accepting the kinetic power $\dot{K}$ in the conventional form, then,

$$\dot{K} = \int_v \rho_o (\dot{V}^i V_i)\, dv \qquad (1.5\text{-}1b)$$

Restricting the derivation for the situation where nonmechanical power supply will be only from thermal considerations and there will be no multipolar heat flux, the supply of thermal energy per unit time by heat flux h across the surface and by distributed energy sources in the body

per unit mass q, is given by,

$$\sum_{\alpha} F_{\alpha} = \int_{v} \rho_o \, q \, dv + \int_{s} h ds \tag{1.5-1c}$$

Heat energy h will be absorbed by the material and will be supplied by radiation from the external system across the surface s. Further, let θ be a local absolute temperature and let Q^i be the conventional heat flux vector through the surface s into the volume v. Substituting expressions for K, P and $\sum_{\alpha} F_{\alpha}$ in the balance equation (1.4-2) and after some rearrangements of terms, it yields,

$$\int_{v} \rho_o (\dot{V}^i V_i + \dot{e}) \, dv = \int_{v} \rho_o (q + F^i V_i + F^{ij} V_{j|i}) \, dv + \int_{s} (h + P^i V_i + P^{ij} V_{j|i}) \, ds \tag{1.5-1}$$

In the next step, relations between the surface forces and stresses together with the heat flux h and heat flux vector Q^i over the surface s is established. The method adopted here is similar to that given in references[4,5].

Consider the volume v in Eq. (1.5-1) to be an internal tetrahedral element bounded by a plane with arbitrary unit normal n_i and by planes through the point X parallel to the coordinate planes. This element would be subjected to the internal stresses T^{ij} and T^{ijk} as well as heat flux vector Q^i and external surface loads P^i, P^{ij} ~~and external~~ heat flux h. If ds is the area of the plane of the tetrahedron normal

to n_i and ds_i is the element of area of the plane of the tetrahedron normal to X^i, then,

$$ds_i = n_i ds \tag{1.5-2}$$

If the volume of the tetrahedron shrinks to zero preserving the orientations of its planes, then the volume integrals in Eq. (1.5-1) vanish and hence this equation can now be modified to account for the internal stresses and heat flux vector to the form,

$$\begin{aligned}(P^i - n_j T^{ji}) V_i + (P^{ji} - n_k T^{kji}) V_{i|j} \\ + h - n_i Q^i = 0\end{aligned} \tag{1.5-3}$$

For an arbitrary constant rigid body velocity, stress tensors T^{ji}, T^{jki}, surface loads P^i, P^{ji}, heat flux h and heat flux vector Q^i remain unaltered. Hence,

$$(P^i - n_j T^{ji}) V_i = 0 \tag{1.5-4}$$

where V_i is now the arbitrary rigid body velocity. Hence,

$$P^i = n_j T^{ji}$$

and Eq. (1.5-3) becomes,

$$(P^{ji} - n_k T^{kji}) V_{i|j} + h - n_i Q^i = 0 \tag{1.5-5}$$

Again $V_{i|j}$ can be written as

$$V_{i|j} = A_{ij} + W_{ij} \tag{1.5-6}$$

where the symmetric tensor (deformation rate)

$$A_{ij} = (V_{i|j} + V_{j|i})/2 \tag{1.5-7}$$

and antisymmetric tensor (spinor),

$$W_{ij} = (V_{i|j} - V_{j|i})/2 \tag{1.5-8}$$

Substituting (1.5-6) in the Equation (1.5-5),

$$(P^{ji} - n_k T^{kji})(A_{ij} + W_{ij}) + h - n_i Q^i = 0 \tag{1.5-9}$$

Next applying a rigid body angular velocity and observing that P^{ji}, T^{kji}, h, Q^i and A_{ij} are unaltered but W_{ij} changes to

$$W_{ij} \rightarrow W_{ij} + 2\Omega_{ij} \tag{1.5-10}$$

and Eq. (1.5-9) yields,

$$(P^{ji} - n_k T^{kji})\ \Omega_{ij} = 0 \tag{1.5-11}$$

Since Ω_{ij} is an arbitrary antisymmetric tensor, Eq. (1.5-11) can be written as,

$$P^{ji} - P^{ij} - n_k (T^{kji} - T^{kij}) = 0 \tag{1.5-12}$$

and (1.5-9) reduces to,

$$(P^{ji} - n_k T^{kji})\ A_{ij} + h - n_i Q^i = 0 \tag{1.5-13}$$

If P^{ji} and T^{kji} is split into symmetric and antisymmetric parts, Eq. (1.5-12) and (1.5-13) can be written in a slightly different form. Thus,

$$P^{|ji|} - n_k T^{k|ji|} = 0 \tag{1.5-14}$$

and $(P^{(ji)} - n_k T^{k(ji)})\ A_{ij} + h - n_i Q^i = 0$ (1.5-15)

where the indices enclosing the braces represent the symmetric part and within the square brackets, the antisymmetric

parts. In deriving Eq. (1.5-15), the symmetry criteria of A_{ij} has been taken into consideration.

Now consider materials for which $(P^{(ji)} - n_k T^{k(ji)})$ on the arbitrary boundary surface of the body does not depend explicitly on the velocity gradients and hence does not depend on A_{ij}. Then, since the element of the latter quantity can be chosen arbitrarily and independently of each other, subjected to symmetry restrictions, Eqs. (1.5-14) and (1.5-15) yield,

$$(P^{ji} - n_k T^{kji}) V_{i|j} = 0 \qquad (1.5\text{-}16)$$

i.e., $$P^{ji} = n_k T^{kji}$$

and $$h = n_i Q^i \qquad (1.5\text{-}17)$$

With the help of Eqs. (1.5-4), (1.5-14) and (1.5-17), the balance equation (1.5-1) can be written as,

$$\int_v \rho_o \dot{V}^i V_i \, dv + \int_v \rho_o \dot{e} \, dv = \int_v \rho_o (q + F^i V_i + F^{ij} V_{j|i}) dv$$
$$+ \int_s n_i Q^i \, ds + \int_s (n_k T^{ki} V_i + n_k T^{kij} V_{j|i}) \, ds \qquad (1.5\text{-}18)$$

By transforming surface integrals to volume integrals by usual way after making appropriate smoothness assumptions, Eq. (1.5-18) can be modified to,

$$\int_v \rho_o \dot{V}^i V_i \, dv + \int_v \rho_o \dot{e} \, dv = \int_v (\rho_o q + Q^i{}_{|i} + \rho_o F^i V_i$$
$$+ \rho_o F^{ki} V_{i|k} + T^{ki}{}_{|k} V_i + T^{ki} V_{i|k}$$
$$+ T^{kji}{}_{|k} V_{i|j} + T^{kji} V_{i|jk}) \, dv \qquad (1.5\text{-}19)$$

After some rearrangement of terms, Eq. (1.5-19) yields,

$$\int_V (T^{ki}{}_{|k} + \rho_o F^i - \rho_o \dot{V}^i) V_i \, dv + \int_V | (T^{kji}{}_{|k} + \rho_o F^{ji} + T^{ji}) \, V_{i|j} - \rho_o \dot{e} + \rho_o q + Q^i{}_{|i} + T^{kji} \, V_{i|jk} | \, dv = 0 \qquad (1.5\text{-}20)$$

This balance equation should be invariant under a change of rigid body constant velocity. Since due to this change, in addition to T^{ki}, T^{kji}, q and Q^i, $\dot{e}$, F^i and F^{ki} also remain invariant by uniform rigid body velocity, Eq. (1.5-20) reduces to,

$$\int_V (T^{ki}{}_{|k} + \rho_o F^i - \rho_o \dot{V}^i) \, V_i \, dv = 0 \qquad (1.5\text{-}21)$$

and

$$\int_V \rho_o \dot{e} \, dv = \int_V (t^{*ji} \, V_{i|j} + T^{kji} \, V_{i|jk} + \rho_o q + Q^i{}_{|i}) \, dv \qquad (1.5\text{-}22)$$

where, $$t^{*ji} = T^{kji}{}_{|k} + \rho_o F^{ji} + T^{ji} \qquad (1.5\text{-}23)$$

Since this is true for any arbitrary volume, dropping the integral, the equation may be written as,

$$T^{ki}{}_{|k} + \rho_o F^i - \rho_o \dot{V}^i = 0 \qquad (1.5\text{-}24)$$

and $$\rho_o \dot{e} = t^{*ji} \, V_{i|j} + T^{kji} \, V_{i|jk} + \rho_o q + Q^i{}_{|i} \qquad (1.5\text{-}25)$$

Again, if additional assumptions are made that $\dot{e}$, F^{ji} and q remain unaltered by superposed uniform rigid body angular

velocity, the body occupying the same position at time t, then,

$$t^{*|ji|} \quad W_{ij} = 0 \tag{1.5-26}$$

This shows that the antisymmetric part of t^{*ji} does not produce any work. Thus Eq. (1.5-25) will yield,

$$\rho_o \dot{e} = t^{*(ji)} A_{ij} + T^{(kj)} V_{i|jk} + \rho_o q + Q^i{}_{|i} \tag{1.5-27}$$

Eq. (1.5-24) furnishes the classical equation of motion derived from Cauchy's first law[3], i.e., momentum balance principle and Eq. (1.5-26), which being true for any arbitrary W_{ij}, reduces to,

$$t^{*|ji|} = 0 \tag{1.5-28}$$

provides moment of momentum balance equation, i.e., Cauchy's second law of motion. At this stage, no further progress is possible unless the independent variables in e, i.e., ν_α are assumed explicitly.

1.6 Internal Energy:

In classical theory, the elastic energy, which is the internal energy for an elastic medium in isothermal condition, is given by the form,

$$E = \int_V e' \, dv$$

where e' is the energy of deformation per unit of reference volume. Now e' at any point X and at time t is determined by

instantaneous configuration $\bar{x}_t(\bar{X})$ of an arbitrary small neighbourhood of point $\tilde{N}(\bar{X})$ containing X. Suppose that $\bar{x}_t(\bar{X})$ has p number of derivatives in that neighbourhood. Then the relative position vector of the point X and any other point X' in $\tilde{N}(\bar{X})$ can be written using Taylor's series[9], in the form,

$$x^i(\bar{X}') - x^i(\bar{X}) = x^i{}_{|j}\ (\bar{X})\ dX^j + \frac{1}{2!}\, x^i{}_{|jk}\ (\bar{X})dX^j dX^k$$

$$+ \ldots + \frac{1}{p!}\, x^i{}_{|i_1 i_2 \ldots i_p}\ (\bar{X})dX^{i_1}dX^{i_2}\ldots dX^{i_p} + O(\delta^{p+1})$$

where $O(\delta^{p+1})$ represent a term of order p+1 in the diameter δ of $\tilde{N}(\bar{X})$. In classical theory the energy term is given by,

$$e' = e'\ (x^i{}_{|j},\ \bar{X}) \tag{1.6-1}$$

The obvious and natural generalization of (1.6-1) is,

$$E = \int_v e'(x^i{}_{|j},\ x^i{}_{|jk},\ \ldots,\ x^i{}_{|i_1 i_2 \ldots i_N},\ \bar{X})dv$$

Following Noll's[10] terminology, if N is the order of highest gradient present in the energy density function, the material may be termed as grade N material and when N is greater than one, it is nonsimple material. In the light of this definition, the materials referred in the classical theory and represented by the Eq. (1.6-1) are simple materials of grade one.

The Eq. (1.5-25) which gives the rate of increment of internal energy has second derivatives as the highest velocity gradient which has actually followed from the assumption that only dipolar forces are present in the continuum. It

automatically leads to the conclusion that atmost the material may be of grade two. Thus in a reverse way, the assumption of grade two material will necessitate the presence of dipolar forces and any higher order forces cannot produce work and hence can be completely disregarded. In this situation, apart from the variable η which is a scalar, ν_α should span the whole space upto second derivatives of the displacement functions. Hence, the specified internal energy can be written as,

$$e = e\,(\eta\,,\ x^i{}_{|j},\ x^i{}_{|jk},\ \bar{X}) \qquad (1.6\text{-}2)$$

But this expression will have to be restricted so that the scalar function e will be invariant under rigid body rotation. There is no unique way of representing this function in invariant form. However, Toupin[9] has shown that all these conditions will be satisfied if it is assumed that,

$$e = e\,(\eta\,,\ E_{ij},\ E_{ijk},\ \bar{X}) \qquad (1.6\text{-}3)$$

where E_{ij} is classical strain tensor and

$$E_{ijk} = E_{ij|k} + E_{ik|j} - E_{jk|i} \qquad (1.6\text{-}4)$$

The dependence of the density function e on $\bar{X}$ in the Eq.(1.6-3) is only to account the nonhomogeneity of the material. Now, if thermodynamic tensions are designated as,

$$e\hat{T}^{ij} = \rho_o \frac{\partial e}{\partial U_{j|i}}$$

and

$$e\hat{T}^{(kj)i} = \rho_o \frac{\partial e}{\partial U_{i|jk}} \qquad (1.6\text{-}5)$$

then, the time rate of internal energy can be written as,

$$\rho_o \dot{e} = \rho_o \theta \dot{\eta} + {}_e\hat{T}^{ij} V_{j|i} + {}_e\hat{T}^{(kj)i} V_{i|jk} \quad (1.6\text{-}6)$$

Since Eq. (1.6-6) has to be invariant under superposed uniform rigid body angular velocity,

$${}_e\hat{T}^{|ij|} = 0 \quad (1.6\text{-}6a)$$

and $$\rho_o \dot{e} = \rho_o \theta \dot{} + {}_e\hat{T}^{(ij)} A_{ji} + {}_e\hat{T}^{(kj)i} V_{i|jk} \quad (1.6\text{-}6b)$$

To get a more clear picture regarding the nature of the function η, eliminating $\dot{e}$ from the Eqs. (1.5-27) and (1.6-6b), the expression for the rate of entropy production can be written as,

$$\begin{aligned}\rho_o \theta \dot{\eta} &= (t^{*(ij)} - {}_e\hat{T}^{(ij)}) A_{ji} \\ &+ (T^{(kj)i} - {}_e\hat{T}^{(kj)i}) V_{i|jk} \quad (1.6\text{-}7) \\ &+ \rho_o q + Q^i{}_{|i}\end{aligned}$$

Since thermodynamic tensions and recoverable, the production of entropy, is due to excess of mechanical work over the recoverable parts and nonmechanical supply of energy. Designating,

$${}_d\hat{T}^{(ij)} = t^{*(ij)} - {}_e\hat{T}^{(ij)}$$

and $${}_d\hat{T}^{(kj)i} = T^{(kj)i} - {}_e\hat{T}^{(kj)i} \quad (1.6\text{-}8)$$

Eq. (1.6-7) becomes,

$$\rho_o \theta \dot{\eta} = {}_d\hat{T}^{(ij)} A_{ji} + {}_d\hat{T}^{(kj)i} V_{i|jk} + \rho_o q + Q^i{}_{|i} \quad (1.6\text{-}9)$$

The prefix 'd' is used to denote that the quantities are of dissipative nature, i.e., are not recoverable.

Since entropy η depends upon thermodynamic path, it is possible to get its constitutive equation which will be a function of temperature θ, mono and dipolar displacement gradients and velocity gradient upto a certain order (with respect to time, e.g., rth order dipolar velocity gradient is $\partial^r U_{i|jk}/\partial t^r$) and the past history of the thermodynamic path. The main point is that the dissipative stresses and hyperstresses can be obtained from the constitutive equation for η provided the entire past thermodynamic path is known.

Substituting Eq. (1.6-9) into (1.6-6b), internal energy expression can be written as,

$$\rho_o \dot{e} = \hat{T}^{(ij)} A_{ji} + \hat{T}^{(kj)i} V_{i|jk} + \rho_o q + Q^i{}_{|i} \qquad (1.6\text{-}10)$$

where $\hat{T}^{(ij)} = {}_e\hat{T}^{(ij)} + {}_d\hat{T}^{(ij)}$ (1.6-11)

and $\hat{T}^{(kj)i} = {}_e\hat{T}^{(kj)i} + {}_d\hat{T}^{(kj)i}$

Comparing Eq. (1.6-10) and (1.5-25) and recalling that the first and second order velocity gradients are independent kinematic variables, it can be concluded that,

$$(t^{*(ij)} - \hat{T}^{(ij)}) A_{ji} = 0$$

and $$(T^{(kj)i} - \hat{T}^{(kj)i}) V_{i|jk} = 0 \qquad (1.6\text{-}12)$$

and consequently,

$$t^{*(ij)} = \hat{T}^{(ij)} \quad \text{and} \quad T^{(kj)i} = \hat{T}^{(kj)i} \qquad (1.6\text{-}13)$$

To reduce the number of variables, symmetric parts of the dipolar stresses, $T^{(kj)i}$, may be replaced by $\hat{T}^{(kj)i}$ from $(1.6\text{-}13)_2$, when using Eqs. (1.5-28) and $(1.6\text{-}13)_1$, the symmetric and antisymmetric parts of the monopolar stresses can be written as,

$$T^{(ji)} = \hat{T}^{(ji)} - \hat{T}^{(k(j)i)}{}_{|k} - \rho_o\, F^{(ji)} - T^{|k(j|i)}{}_{|k} \tag{1.6-14}$$

and
$$T^{|ji|} = -\hat{T}^{(k|j)i|}{}_{|k} - \rho_o F^{|ji|} - T^{|k|j|i|}{}_{|k} \tag{1.6-15}$$

In the next section, it will be shown that all the unknown variables involved in the governing differential equations (1.5-24), (1.6-14) and (1.6-15) are not determinable independently, i.e., the set of differential equations do not form a determinate set.

1.7 Indeterminancy of Stresses:

Apart from temperature θ, the total number of unknowns in the system is twentyone. This consists of nine T^{ji}, nine $T^{|kj|i}$ and three displacements U_i. The symmetric parts of T^{kji}, i.e., $T^{(kj)i}$ are no longer independent unknowns, since they have been replaced by $\hat{T}^{(kj)i}$ from constitutive equations, which are functions of displacements. On the other hand, the number of governing equations are twelve which are three from (1.5-24) and nine from (1.5-14) and (1.5-15). Hence total system in mechanical parts is not determinate.

If the expressions for $T^{(ji)}$ and $T^{|ji|}$ from (1.6-14) and (1.6-15) are substituted in the equation of motion (1.5-24), the terms $T^{|k(j|i)}{}_{|jk}$ and $T^{|k|j|i|}{}_{|jk}$ vanish because of the antisymmetry of the indices j and k where as the covariant differentiation with respect to j and k is independent of order and hence are symmetric. Again work rate due to this part of stress,

$$T^{|kj|i} \cdot V_{i|jk}$$

is also zero. Although, this part does not contribute to the equation of motion, nor produces any work, it plays important role for the boundary conditions[4,5]. For the sake of simplicity, this part will be assumed to be totally nonexistent within the body and also on the boundary.

With this limitation, the governing equations may be collected together and written in the integral form as follows:

$$\int_V (T^{ji}{}_{|j} + \rho_o F^i - \rho_o \dot{V}^i) V_i \, dv = 0 \qquad (1.7\text{-}1)$$

$$\int_V (T^{(ji)} - \hat{T}^{(ji)} + \hat{T}^{(k(j)i)}{}_{|k} + \rho_o F^{(ji)}) A_{ij} \, dv = 0 \qquad (1.7\text{-}2)$$

$$\int_V (T^{|ji|} + \hat{T}^{(k|j)i|}{}_{|k} + \rho_o F^{|ji|}) W_{ij} \, dv = 0 \qquad (1.7\text{-}3)$$

The corresponding conditions to be satisfied on the boundaries

are,

$$\int_s (P^i - n_j\, T^{ji})\, V_i\, ds = 0$$

$$\int_s (P^{(ji)} - n_k\, T^{(k(j)i)})\quad A_{ij}\, ds = 0 \qquad (1.7\text{-}4)$$

and $$\int_s (P^{|ji|} - n_k\, T^{(k|j)i|})\, W_{ij}\, ds = 0$$

Combining Eq. (1.7-1) with $(1.7\text{-}4)_1$, the overall momentum balance over an arbitrary small domain can be written as,

$$\int_v (T^{ji}{}_{|j} + \rho_o F^i - \rho_o \dot{V}^i)\, V_i\, dv$$
$$+ \int_s (P^i - n_j\, T^{ji})\, V_i\, ds = 0 \qquad (1.7\text{-}5)$$

Now, since Eq. (1.7-2) and (1.7-3) do not contribute any **new development, it is advantageous to combine them together** with $(1.7\text{-}4)_2$ and $(1.7\text{-}4)_3$ to yield,

$$\int_v (T^{ji} - \hat{T}^{(ji)} + \hat{T}^{(kj)i}{}_{|k} + \rho_o\, F^{ji})\, V_{i|j}\, dv$$

$$+ \int_s (P^{ji} - n_k \hat{T}^{(kj)i})\, V_{i|j}\, ds = 0 \qquad (1.7\text{-}6)$$

In these equations, F^i and F^{ji} are body forces, T^{ji} are monopolar unknown stresses, P^i and P^{ji} are surface forces and V_i are unknown velocities. All the above quantities are functions of space and time. Moreover, $\hat{T}^{(ji)}$ and $\hat{T}^{(kj)i}$ are thermodynamic stresses supplied from constitutive equations.

1.8 Equation for Thermal State:

The equation for thermal state can be easily obtained by substituting Eq. (1.4-15) in (1.6-9) for $\dot{\eta}$, i.e.,

$$\rho_o \, \theta \frac{d}{dt} (\psi , \theta) + \dot{\rho}_o \, q + Q^i{}_{|i} + w_d = 0 \qquad (1.8\text{-}1)$$

where

$$w_d = {}_d\hat{T}^{(ji)} A_{ij} + {}_d\hat{T}^{(kj)i} V_{i|jk} \qquad (1.8\text{-}2)$$

and represents the dissipative work done. Since,

$$(\dot{\theta} \, Q^i)_{|i} = \dot{\theta}_{|i} \, Q^i + \dot{\theta}\theta^i{}_{|i} \qquad (1.8\text{-}3)$$

multiply Eq. (1.8-1) by $\dot{\theta}$ and then substituting the expression for $\dot{\theta}Q^i{}_{|i}$ from (1.8-3)

$$\rho_o \, \dot{\theta}\theta \frac{d}{dt} (\psi , \theta) + \rho_o \, q \, \dot{\theta} - \dot{\theta}_{|i} \, Q^i + \dot{\theta} \, w_d + (\dot{\theta}Q^i)_{|i} = 0 \qquad (1.8\text{-}4)$$

Since this is true for the whole volume or a part there of, integrating the expression over the volume v and then transforming the volume integral of the term $(\dot{\theta}Q^i)_{|i}$ to the surface by Green-Gauss theorem, Eq. (1.8-4) yields,

$$\int_v (\rho_o \, \dot{\theta}\theta \frac{d}{dt} (\psi , \theta) + \rho_o \, \dot{\theta} \, q - \dot{\theta}_{|i} Q^i + w_d \dot{\theta}) \, dv + \int_s n_i \, \dot{\theta} Q^i \, ds = 0 \qquad (1.8\text{-}5)$$

which is to be associated with the relation on the boundary surface,

$$h = n_i Q^i \qquad (1.8\text{-}6)$$

Here s is the boundary surface enclosing the volume v and n_i is the unit outward normal at a point X. In Eq. (1.8-5), θ is temperature, Q^i is heat flux vector, q is the distributed energy source inside the body and ψ is free energy and in (1.8-6), h is heat flux. Constitutive equations are to be supplied for ψ and Q^i Eq. (1.8-5) forms the basis equation for thermal state in integrodifferential form. It will be kept as it is for later derivation.

1.9 Constitutive Equations:

The general physical laws in themselves do not suffice to determine the deformation of motion of a body subjected to given loading. Before a determinate problem can be formulated, it is usually necessary to specify the material of which the body is made. In the discipline of continuum mechanics such prescriptions are designated as constitutive equations. The derivation of constitutive equations is a fundamental issue in the theoretical development of nonlinear mechanics. The basic derivations though based on much mathematical rigour, are not unambiguous regarding their fundamental assumptions[11,12,20]. Moreover, from practical point of view, the constitutive equations are not exact descriptions of even the gross physical properties of real materials. The best that can be said of any mathematical model of material behavior is that it provides a useful description of certain features of the behavior of some real materials.

The concepts of irreversible thermodynamics were applied to the study of linear materials with memory by Biot[13] and Ziegler[14]. These applications have been extended to include nonlinear materials undergoing large deformations by Valanis[15], Coleman and Gurtin[16], and Coleman and Noll[17]. Coleman and Gurtin[16] have pointed out that this approach to continuum thermodynamics is but one of several other approaches including those based on constitutive equations of differential type and on the axiom of fading memory. The differential type constitutive equations have been studied by Coleman and Mizel[18], Mandel and Brun[21] and others. On the other hand, the axiom of fading memory was developed by Green and Rivlin[22], Noll[23] and Coleman[24] and elaborated by Coleman and Mizel[25], Mizel and Wang[26] and others. The equivalence of these two approaches, in certain special forms, has been studied by Coleman and Noll[27], Lubliner[28] and Lianis[29].

Constitutive equations prescribed in the above literatures are exceedingly complex and general. These equations are not usually suitable for use, in their full generality, in the solution of initial and boundary value problems. The purpose of such equations is to provide a framework within which observed behavior under a wide range of conditions can be correlated, and to serve as sources of simpler equations valid under more narrowly prescribed conditions. Within

such a framework various types of seemingly contradictory special equations can be reconciled[30], as narrow range approximate descriptions. However, in this variant not much of stress will be laid on the mathematical basis for the derivation of the constitutive equations and on their restrictions and the physical significance; on the other hand, fairly general forms of stress, strain and temperature laws will be prescribed which can be simplified for particular cases. However, various special forms of constitutive equations have been listed in the reference.

The thermodynamic stresses $\hat{T}^{(ji)}$ and dipolar stresses $\hat{T}^{(kj)i}$ in the governing equations (1.7-5) and (1.7-6) are of two parts; elastic or recoverable and dissipative or irrecoverable. The principle requirement is that they should not violate the laws of thermodynamics and principle of material frame indifference. Recalling that $\hat{T}^{|ji|}$ is identically zero, $\hat{T}^{|kj|i}$ has been assumed to be nonexistant and that entropy can always be written as a function (or more generally as a functional) of the histories of θ, E_{ij} and E_{ijk} and their time derivatives upto the present time, the following constitutive equations can be prescribed:

$$\hat{T}^{(ji)} = \hat{\hat{T}}^{(pq)}(E_{kl},\ E_{klm},\ \mathbf{A}_{kl},\ \mathbf{A}_{klm},\ \theta,\ \dot{\theta},\ t^p)\left(\frac{\partial \mathbf{E}_{pq}}{\partial U_{i|j}} + \frac{\partial E_{pq}}{\partial U_{j|i}}\right)/2 \tag{1.9-1}$$

$$\hat{T}^{(kj)i} = \hat{\hat{T}}^{(pq)r}(E_{kl},\ E_{klm},\ A_{kl},\ \mathbf{A}_{klm},\ \theta,\ \dot{\theta},\ t^p)\left(\frac{\partial E_{rqp}}{\partial U_{i|kj}}\right)$$

$$ {}_d\hat{T}^{(ji)} = {}_d\hat{T}^{(pq)}\,(E_{kl},\ E_{klm},\ A_{kl},\ A_{klm},\ \theta,\ \dot{\theta},\ t^p)\left(\frac{\partial E_{pq}}{\partial U_{i|j}} + \frac{\partial E_{pq}}{\partial U_{j|i}}\right)/2 $$

$$ (1.9\text{-}2) $$

$$ {}_d\hat{T}^{(kj)i} = {}_d\hat{\hat{T}}^{(pq)r}\,(E_{kl},\ E_{klm},\ A_{kl},\ A_{klm},\ \theta,\ \dot{\theta},\ t^p)\,\frac{\partial E_{rqp}}{\partial U_{i|kj}} $$

In relations (1.9-1) and (1.9-2) the argument t^p denotes the dependence of the functions on the history of their arguments for $t \geq t^p > -\infty$, where t is the present value of time. The arguments A's are Rivlin-Ericksen tensor[3]. For a general case, these tensors are given by,

$$ \overset{(r)}{A}_{ij} = \sum_{\alpha=0}^{r} \binom{r}{\alpha} G^{mn}\,\overset{(\alpha)}{V}_{m|i}\,\overset{(r-\alpha)}{V}_{n|j} \qquad (1.9\text{-}3) $$

and $$ \overset{(r)}{A}_{ijk} = \sum_{\alpha=1}^{r} \binom{r}{\alpha} G^{mn}\,\overset{(r-\alpha)}{V}_{m|i}\,\overset{(\alpha)}{V}_{n|jk} $$

with $\overset{(0)}{V}_{m|i} = G_{mi}$. $\overset{(1)}{A}_{ij} = A_{ij}$ and $\overset{(1)}{A}_{ijk} = A_{ijk}$

In (1.9-3), the braces are permutation symbols

$$ \binom{r}{\alpha} = \frac{r!}{\alpha!\,(r-\alpha)!} $$

and $\overset{(r)}{V}_{m|i}$ and $\overset{(r)}{V}_{m|jk}$ are higher order velocity gradients i.e.,

$$ \overset{(r)}{V}_{m|i} = \frac{D^r(U_{m|i})}{Dt^r} $$

and $$ \overset{(r)}{V}_{m|jk} = \frac{D^r(U_{m|jk})}{Dt^r} $$

In constitutive equations (1.9-1) and (1.9-2) only the velocity gradients upto one (r = 1) has been considered directly

and any higher order effect can be incorporated through the definition of the functional $\hat{\hat{T}}(.)$ and its histories. Since, in the next chapter, an incremental approach will be adopted for solution which will actually move through the range of the parameter (history), no special consideration need to be attributed to these functionals (for an elegent proof see reference 31).

For example consider the stress strain law of a visco-elastic material formulated on the basis of n step history[22]. Since an arbitrary history can be approximated as closely as desired by a step history, the stress components which can be written in the form,

$$\hat{\hat{T}}^{(pq)}(t) = \sum_{n=1}^{\infty} T_n^{(pq)}(t) \tag{1.9-4}$$

will give the exact stress for any history, assuming reasonable smoothness of dependence of stress on the strain history. Pipkin and Rogers[32] have taken $\hat{T}_n^{(pq)}(t)$ in the form of multiple integrals which, in present notations for three dimensional case can be shown to be,

$$\hat{T}_n^{(pq)}(t) = \frac{1}{n!} \int_{-\infty}^{t} \ldots \int_{-\infty}^{t} \delta_1' \ldots \delta_n' \, R_n^{(pq)}(E_{ij}(t_1^*), t - t_1^* \ldots, E_{ij}(t_n^*), t - t_n^*) \tag{1.9-5}$$

where,

$$\delta_k' \, R_n^{(pq)}(.) = \frac{\partial R_n^{(pq)}(.)}{\partial E_{rs}} \, \dot{E}_{rs}(t^*_k) \, dt^*_k$$
$$k = 1, 2, \ldots, n \tag{1.9-6}$$

For one step method, it takes the simplified familiar form,

$$\hat{T}_1^{(pq)}(t) = \int_{-\infty}^{t} \frac{\partial R_1^{(pq)}(E_{ij}(t_1^*), t-t_1^*)}{\partial E_{rs}} \dot{E}_{rs}(t_1^*)\, dt_1^* \tag{1.9-7}$$

Again consider a non-viscous elastic plastic medium having no memory (a more general case for a medium with memory has been treated by Kluitenberg[33]). In this case, it is possible to introduce the physical assumption that if plasticity phenomena occur, it will be governed by the yield function Φ, defined in cartesian coordinate system, by

$$\Phi = \frac{1}{2} \sum_{i,j=1}^{3} (\hat{T}^{(ij)})^2 - k^2 \tag{1.9-8}$$

which is Von-Mises yield function. The associated Von-Mises flow law can be expressed as,

$$\frac{dE'_{ij}}{dt} = b^* \frac{\partial \Phi}{\partial T^{(ij)}} \tag{1.9-9}$$

where b^* vanishes for $\Phi < 0$. For such substance, the stress strain relation is given by

$$ab^* \hat{T}^{(ij)} + \frac{dT^{(ij)}}{dt} = a \frac{dF_{ij}}{dt} \tag{1.9-10}$$

where a is a material constant and b^* vanishes for $\Phi < 0$. This is Prandtl-Reuss equation.

In the equation for thermal state (1.8-5), the constitutive unknowns are ψ and Q^i. From the similarity of

expressions (1.7-3), (1.5-15) and (1.5-10), the functional form of the Helmholtz free energy, ψ can be assumed as,

$$\rho_o \psi = \hat{\psi} (E_{ij}, E_{ijk}, A_{ij}, A_{ijk}, \theta, \dot{\theta}, t^p) \qquad (1.9\text{-}11)$$

which for a linear elastic material, can be explicitly written as,

$$\rho_o \psi = \frac{1}{2} C^{ijkl} E_{ij} E_{kl} + \underset{\sim}{C}^{ij} E_{ij} \theta + \frac{1}{2} \underline{\bar{C}} \theta^2 \qquad (1.9\text{-}12)$$

where C^{ijkl}, $\underset{\sim}{C}^{ij}$ and $\underline{\bar{C}}$ are material coefficients. For linearly elastic case, these constants may be assumed to be the function of θ. For nonlinear material they can be considered as function of θ as well as E_{ij}. The coefficients C^{ijkl} and $\underset{\sim}{C}^{ij}$, have the symmetry restrictions,

$$C^{ijkl} = C^{jikl} = C^{ijlk} = C^{klij}$$

$$\text{and} \quad \underset{\sim}{C}^{ij} = \underset{\sim}{C}^{ji} \qquad (1.9\text{-}13)$$

As regard the heat flux vector Q^i, a functional constitutive equation can be written in the form,

$$Q^i = \hat{Q}^i (E_{kl}, E_{klm}, \theta_{|k}, t^p) \qquad (1.9\text{-}14)$$

Coleman et al.[34] have proved through the restriction of second law of thermodynamics that for a rigid heat conductor, the classical Fourier's law of heat conduction can be obtained as a first order approximation having the form,

$$\begin{aligned} Q^i &= K^{ij} Q_{|j} \\ K^{ij} &= K^{ij}(\theta) \end{aligned} \qquad (1.9\text{-}15)$$

1.10 Summary:

1.10.1 Assumptions:

The following assumptions have been made in the preceding sections for deriving the general governing equations for continuum.

1. The material will obey conservation of mass and first and second laws of thermodynamics.
2. Hypothesis of material frame indifference is applicable.
3. Concept of force defined through work rate principle is valid.
4. Displacement derivatives upto second order will be active for the internal energy expression.
5. Supply of nonmechanical energy is only due to thermal effect.
6. Multipolar kinetic energy and heat flux vector has been neglected.
7. Antisymmetric parts of the dipolar stresses, ~~x~~i.e., $T^{|kj|i}$ are nonexistent.

1.10.2 Governing Equations and Constitutive Relations:

The governing equations for the continuum can be rewritten from Eqs. (1.7-20), (1.7-21) and (1.8-5) for mechanical effect as,

$$\int_V (T^{ji}{}_{|j} + \rho_o F^i - \rho_o \dot{V}^i)\, V_i \, dv + \int_S (P^i - n_j T^{ji}) Vi \, ds = 0 \qquad (1.10\text{-}1)$$

and $$\int_v (T^{ji} - \hat{T}^{(ji)} + \hat{T}^{(kj)i}{}_{|k} + \rho_o F^{ji})\, V_{i|j}\, dv + \int_s (P^{ji} - n_k\, T^{(kj)i})\, V_{i|j}\, ds = 0 \qquad (1.10\text{-}2)$$

and for thermal effect as,

$$\int_v (\rho_o\, \dot{\theta}\, \theta\, \frac{d}{dt}(\psi, \theta) + \rho_o\, \dot{\theta} q - \dot{\theta}_{|i}\, Q^i + w_d \dot{\theta})\, dv + \int_s n_i\, \dot{\theta}\, Q^i\, ds = 0 \qquad (1.10\text{-}3)$$

where $$w_d = {}_d\hat{T}^{(ji)}\, A_{ij} + {}_d\hat{T}^{(kj)i}\, V_{i|jk} \qquad (1.10\text{-}4)$$

Here v is any arbitrary volume bounded by the surface s having an outward normal n_i at a point X. T^{ji}, F^i, F^{ji}, P^i and P^{ji} in (1.10-1) and (1.10-2) are respectively monopolar stresses, mono and dipolar body and surface forces. $\hat{T}^{(ji)}$ and $\hat{T}^{(kj)i}$ are thermodynamic tensions and V_i is the velocity. In (1.10-3), θ, q and Q^i are respectively temperature, distributed heat source and heat flux vector. The symbols w_d and ρ_o stand for dissipative work rate and density of mass, while ψ denotes free energy. All the quantities referred here are in terms of reference configuration. The heat equation (1.10-3) must be complemented with the relation,

$$h = n_i\, Q^i \qquad (1.10\text{-}5)$$

on the surface s, where h is the heat flux over the surface.

Specification of mechanical and thermal properties of matter through constitutive relations for mono and dipolar stresses, free energy and heat flux vector may be prescribed through any general functionals provided they satisfy frame

indifference principle, laws of thermodynamics specially second law restrictions and any other physical characteristics imparted on the material properties. Also, as per assumption, they are not supposed to contain derivatives of displacements of order more than two. In restricted sense, they are prescribed by Eqs. (1.9-1) to (1.9-4) and are given as follows:

$$\hat{T}^{(ji)} = \hat{\hat{T}}^{(pq)} (E_{lm}, E_{lmn}, A_{lm}, A_{lmn}, \theta, \dot{\theta}, t^p) \left(\frac{\partial E_{pq}}{\partial U_{i|j}} + \frac{\partial E_{pq}}{\partial U_{j|i}}\right)/2$$

$$_d\hat{T}^{(ji)} = {_d\hat{\hat{T}}}^{(pq)} (E_{lm}, E_{lmn}, A_{lm}, A_{lmn}, \theta, \dot{\theta}, t^p) \left(\frac{\partial E_{pq}}{\partial U_{i|j}} + \frac{\partial E_{pq}}{\partial U_{j|i}}\right)/2$$

$$\hat{T}^{(kj)i} = \hat{\hat{T}}^{(pq)r} (E_{lm}, E_{lmn}, A_{lm}, A_{lmn}, \theta, \dot{\theta}, t^p) \frac{\partial E_{rqp}}{\partial U_{i|kj}} \tag{1.10-6}$$

$$_d\hat{T}^{(kj)i} = \hat{\hat{T}}^{(pq)r} (E_{lm}, E_{lmn}, A_{lm}, A_{lmn}, \theta, \dot{\theta}, t^p) \frac{\partial E_{rqp}}{\partial U_{i|kj}}$$

The constitutive relation for free energy ψ has been set as functional,

$$\psi = \hat{\hat{\psi}} (E_{lm}, E_{lmn}, A_{lm}, A_{lmn}, \theta, \dot{\theta}, t^p) \tag{1.10-7}$$

and heat flux vector, in a simpler form, as

$$Q^i = Q^i (E_{lm}, E_{lmn}, \theta_{|k}, t^p) \tag{1.10-8}$$

In the above equations, E_{lm}, E_{lmn}, A_{lm} and A_{lmn} may be chosen as any strain measures, but conforming to tradition, only classical measures have been quoted here.

$$2E_{lm} = U_{l|m} + U_{m|l} + G^{pq}U_{p|l}\,U_{q|m} \qquad (1.10\text{-}9)$$

$$E_{lmn} = E_{lm|n} + E_{ln|m} - E_{mn|l} \qquad (1.10\text{-}10)$$

With slight modification, Λ_{lm} and Λ_{lmn} can be written in the form,

$$2\Lambda_{lm} = V_{l|m} + V_{m|l} \qquad (1.10\text{-}11)$$

and $$\Lambda_{lmn} = V_{l|mn} \qquad (1.10\text{-}12)$$

The symbol t^p has been introduced to show the dependence of the functional on the past history of the arguments. Hence, if t is the present time, then $t \geq t^p > -\infty$.

In the next chapter, finite element method has been discussed and the above set equations have been transformed for finite element use.

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CHAPTER 2

DISCRETIZATION AND LINEARISATION

2.1 Introduction:

The abysmal mathematical complexities encountered for the exact solution of differential equations for continuum mechanics are virtually insurmountable except for very few idealized cases[1]. Even the classical approximate processes such as asymptotic expansions and weighted residual techniques can be applied to relatively simpler problems[2]. In the past few decades, due to the introduction of high speed computers, some classical approximate methods such as Ritz method, finite difference technique, weighted residual method have been revived and extended. Development of new concepts like invariant imbedding, finite element and other numerical procedures have also been introduced.

Although, finite difference method has been mathematically studied rather well, there are inherent difficulties associated with its application in continuum mechanics problems. The inability to express some of the boundary conditions and the unaccommodative nature of the method for irregular changes of geometry and material properties are some of these difficulties. For the method of invariant imbedding[3,4], which is an extension of Newton-Raphson technique for functional analysis, the disadvantages are difficult

formulation and instead of one original problem a family of problems must be solved resulting in more computational effort. Among the approximate techniques, the finite element method can be considered as one of the most powerful and versatile discretization presently available for the numerical matrix solution of complex structural problems using digital computers.

In conventional matrix structural analysis, the system is usually composed of a finite number of structural elements connected only at a finite number of nodes. The stiffness (in some general sense) properties of the elemental subdivisions can be evaluated exactly within the scope of the adopted theory. Depending on the variables used in the definition of the stiffness matrix, the method can be classified under three heads - (a) displacement method, (b) force method and (c) mixed method.

In finite element method, the domain of the continuum is discretized into a number of disjoint subdomains called elements represented by a set of points which are called nodes. The elements may not only be connected at the nodes, but also along the interelement boundaries. Then the unknown field variables in the integrand of the functional integral representing the equation of motion of the system, is approximated by a set of assumed functions which are expressed in terms of the field variables at the nodes by suitable interpolation

formulas. The stiffness (in some general sense) properties of such elements can now be defined approximately and can be determined in many alternative ways. By far the most popular method for obtaining finite element equations is based on variational principles with respect to minimization of potential or complementary energy or Reissner's energy formulation. However, there is no potential barrier of applying other approximate procedures such as Galerkin's technique or any other principles in mathematical physics.

In the problems of continuum mechanics, the finite element method is easily adaptable to matrix formulation which can be readily analysed by digital computers. The method is capable of approximating quite irregular boundary shapes, complicated boundary conditions and arbitrary, variation of physical and geometrical parameters in its domain of applicability. Here, the algebraic equations resulting from finite element discretization will be, in general, nonlinear and simultaneous. In the absence of any direct procedure for solution, some iterative and/or an incremental technique has to be adopted. The combination of the finite element method with incremental forward integration procedure on the field variables is quite suitable for computer use and general problem solution.

In this chapter, after giving a brief historical review on finite element in section 2 and types of models used for analysis in section 3, different modelling criteria including

the question of convergency is discussed in section 4. The crux of the finite element analysis lies in choosing the distribution of the field variables. In section 5, the field variables have been chosen in a general form without any restriction on the interpolation law and the geometry of the element. In sections 6 and 7, these distributions have been substituted in the equation derived in Chapter 1 to get corresponding equations for finite elements. In section 8, parametric differentiation scheme has been introduced for stepwise solution.

2.2 Historical Review:

The development of discrete method of structural mechanics had its beginning in the early 1950's together with the advent of high speed computers. The concept of finite element was first introduced by Courant[5]. He solved in an approximate manner the classical St. Venant torsion problem by assuming linear distribution of warping function over each triangular domains and then by applying variational principle leading to minimization of potential energy. Prager and Synge[6] and Synge[7] further elaborated the technique by a geometric representation in function space which is often called hypercircle method. Argyris' text[8] for the matrix formulation of the transformation theory of structures is an important milestone in this development work, facilitating a better understanding the structural behaviour and providing a practical and powerful way of employing automatic computation.

This method was first applied to plane problem by Turner and his associates[9] using triangular and rectangular element model. Very soon after its first application, it became evident that the main feature of the displacement consistent finite element is the assumed kinematic field distribution and the method extended to three dimensional analysis[10], plates[11], shells of revolution[12], axisymmetric bodies[13], shallow shell[14], general shell structure[15], eccentrically stiffened cylindrical shell[16] with considerable amount of success in each case. During the past few years, study of the mathematical foundations of the method and different types of variational principles[17] applied for the generation of more general and varied types of finite element models[18,19], as well as its applications[20,21,22] have greatly extended its fields and clarified the basic requirements for its effective formulation.

In early stages of development, a theory establishing necessary and sufficient conditions to guarantee convergence to the true solution was lacking. Such requirements were first considered by Melosh[23]. The development of the stiffness relations may be shown to be equivalent to piecewise Rayleigh-Ritz procedure applied to variational principles, provided continuity requirements at the interfaces of the elements are satisfied. The fundamental considerations for the successful generation of stiffness rest on the choice of

distribution of unknown field variables having the requirements of (a) isotropy, (b) completeness and (c) continuity conditions which have been discussed in references[25,26,27]. The failure of early attempts[29] for the satisfactory construction of triangular elements for plate bending analysis may be attributed due to ignorance of these criteria.

With a firmly established mathematical basis for the generation of finite element stiffness matrices, a systematic development for refinements is possible which may be observed in the 1st[30] and 2nd[31] conference on Matrix Methods in Structural Analysis. The application of such elements to geometrically and physically nonlinear problems which were previously restricted to framed structures, have been possible with considerable confidence and certainty[33,34]. Time dependent phenomena[35,36] and plastic analysis[37,38,39] have also been done employing this technique.

Finally, it should be emphasised that finite element technique is by no means restricted for the solution of structural problems. It can be, as well, applied to any field problems with considerable ease. Zienkiewicz and Cheung[40] have applied this procedure to solve second order problems such as stationary heat flow and torsion, while application to coupled thermoelasticity, nonstationary heat conduction, fluid mechanics problems may be found in reference[31]. It may be noted

here that the references cited above are by no means exhaustive which is mainly because of the ever expanding literature on this subject.

2.3 Types of Model:

Depending upon the characteristics of a finite element model, two principal classes can be defined: (a) displacement model which inherits the chosen displacement field and (b) equilibrium model which starts with the assumption of an equilibrated stress field. Clough, Argyris, Zienkiewicz and their associates are among the notable contributors on displacement model. Veubeke[41] and Morley[42] have applied equilibrium model to plate bending analysis. The displacement and equilibrium models may be considered as subsets of 'mixed model' where both displacements and stresses may be selected independently. Herrmann[43], Prato[44], Dunham and Pister[31] have developed mixed models for various individual problems employing Reissner's variational theory. Again depending upon the method of solution there may be three classes, (a) displacement method, (b) force method and (c) mixed method. However, such classifications are not unambiguous[19]. For example, there may be three variations of finite elements which finally employ nodal displacements as unknowns. The first variation starts by choosing displacement distribution in the individual elements and equations are formulated by application of minimum potential energy

principle. This is a displacement model. The second variation uses an assumed equilibrated stress field and derives the basic equations from the principle of minimum complementary energy. The stress field is subsequently eliminated in terms of conjugate generalised nodal displacements. This may be designated as equilibrium model. The third variation, a hybrid model as has been designated by Pian, is based on modified complementary energy principle, for which compatible displacement functions are assumed along the interelement boundaries, in addition to the assumed equilibrated stress field in each element. The above three variations may be classed as displacement method. Similar subdivisions can also be made in the force method. Hence, a dual hybrid model can also be formulated assuming equilibrated tractions along the interelement boundaries and continuous displacement field in each element. More exhaustive discussions, examples and references are given by Pian and Tong[19] in addition to the associated variational formulations in each case.

2.4 Modelling Criteria:

2.4.1 Efficiency:

An attempt to assess the usefulness of a finite element model must be based on some efficiency criteria. Following are the three criteria[24] that seem to be generally acceptable.

1. Convergence:

As the sizes of elements in a model approach zero, the discretization error in computed field variables, as well as, in geometric simplification, if any, should approach zero. The convergency should be monotonic.

2. Rate of Convergence:

The efficiency of a model should be proportional to its rate of convergence.

3. Computation Efficiency:

The efficiency of a model should be proportional to the ratio of accuracy to computational effort.

Depending upon the above characteristics, there should be some way of expressing the efficiency of a given model from which usability of different models can be compared.

2.4.2 Mathematical Criteria:

It has been already mentioned that the distribution functions of unknown field variables should satisfy the requirements of:

1. Invariance,
2. Completeness and
3. Continuity.

In early stages of development these requirements were not systematically considered which has resulted in many

failures[29,45]. Similar convergency criteria were discussed on heuristic basis by Melosh[23], and Irons[25]. Johnson and Mclay[46], Tong and Pian[47], Walz, Fulton and Cyrus[31] have also investigated convergence in connection with the particular cases of finite element models.Oliveira's works concerns with the mathematical basis[27] and in particular, completeness and convergence criteria[28]. He has derived the sufficient conditions for convergence of a general case of displacement analysis with no particular reference to any definite model. Although, the analysis is restricted to displacement model, it appears that similar conditions can also be applied to any other model.

In classical mathematical formulation considering variational or integral equation approach, it is usually required that the assumed functions should possess derivatives which are continuous upto highest order occurring in the corresponding functionals except possibily at isolated singularities like points, curves or surfaces which necessitate special attentions. In finite element analysis, the distribution functions satisfy the strict continuity requirement within each element, but on the interelement boundaries the admissibility conditions are relaxed in such a way that the functional will be defined for the chosen distributions[19].

1. Invariance: The invariance condition will guarantee that any rigid body parallel shift or rigid body rotation of the

coordinate system will not change the computed field variables except that they are now the components in the new coordinate system. This shows that the chosen distribution should satisfy the invariance requirements given in sec. 3 of Chapter 1. In the case of polynomial distribution function, it is necessary to use complete polynomial upto a certain order, since it is isotropic and obeys the frame indifference principle. In the case of natural coordinate system[26] (which is one having the element interfaces as coordinate surfaces, e.g., area coordinates), the invariance requirement will be automatically satisfied if the distributions are represented by isotropic functions of the natural coordinates[26].

2. Completeness:

The satisfaction of completeness requirement will ensure the convergence of the field variables to the exact values as the element sizes approach zero, provided the continuity conditions are not violated. However, convergence to the true solution may still be achieved even if the continuity conditions are not satisfied, whenever the body force density corresponding to the approximate solutions remains bounded within the element, no matter their size[28].

Completeness of a sequence of families $U_N (N = 1,2,...)$ with respect to a given set $\{\overline{U}_E\}$ is only meaningful provided it is possible to establish a predefined norm relating U_N and each member $\overline{U}_E$ of the set $\{\overline{U}_E\}$. Then the sequence U_N

will be designated as complete, if for a specified $\epsilon > 0$, it is possible to obtain an integer $\bar{N}$ such that there exists a member U_N which satisfies,

$$\| U_N, U_E \| < \epsilon$$

for any $N > \bar{N}$ and for any member U_E of $\{\bar{U}_E\}$ where $\| \cdot , \cdot \|$ is the predefined norm measure.

Unfortunately, it is not always easy, in general, to prove directly the completeness of a distribution function. But as has been mentioned in[26,27], for a polynomial distributi... completeness will be achieved provided,

(a) the distribution function has a number of arbitrary parameters equal to the number of independent unit modes corresponding to the element;

(b) the terms of degree higher than first can be varied independently with respect to the constant term and the coeffi cients which affect linear terms;

(c) the constant terms and the coefficients which affect the linear terms are completely arbitrary.

When using incomplete polynomial expansions in natural coordinates, it is not immediately evident whether the expansion is complete or not. A simple but indirect way has been discussed by Felippa[26].

3. Continuity:

The requisite interelement continuity of chosen field variables, does not affect the convergence of the solution, but it affects the rate of convergence. Lack of continuity may cause non-monotonic convergence of the solution. The method of obtaining the necessary continuity conditions to be satisfied for a particular model have been discussed by Pian[19].

2.4.3 Geometrical Criteria:

In finite element method, discretization is not only done on chosen field variables but also on the continuum itself, i.e., a curved member may be approximated as an assemblage of small triangular flat elements. It has been shown by Walz et al[31] that convergence may not be achieved if a curved structure is discretized by series of straight elements, but the convergence may occur when the element are curved.

Apart from the error resulted from the mathematical or geometrical discretization, there may be round-off error associated with the accuracy with which the numbers are manipulated in a computer. For solution of a problem having a large number of elements, the error resulted from this source may also be critical. However, round-off error is not the characteristic of a finite element model, though

it indirectly effects the solution depending on the order of interpolation formula used to describe the field variables.

In the subsequent sections, finite element equations will be obtained for a general case without restricting to pay particular model.

2.5 Finite Element Derivation:

For the construction of finite element model of the basic field equations (1.10-1), (1.10-2) and (1.10-3) of the last chapter having independent field variable as displacements U_i, stresses T^{ji} and temperature θ, consider the continuum to be represented by R number of elements with a typical element as m and typical node as p and the number of nodes surrounding the node p as M. Let α_1, α_2, α_3 and α_4 be the number of degrees of freedom per element for the interpolation of displacements, gradients of displacements, stresses and temperatures respectively. The interpolating functions of the field variables for a typical element may be approximated as follows:

1. Displacement distribution:

The displacement distribution over a typical element m may be prescribed in terms of α_1 generalized nodal deformations as,

$$U_i\,(X^k,\, t) \;=\; D_{i\alpha}\,(X^k)\; U_{\alpha N}(t) \qquad (2.5\text{-}1)$$
$$\alpha \;=\; 1,\; \ldots,\; \alpha_1$$

where k takes the values of one, two and three representing the three space variables. The generalised deformations need not be nodal displacements only, they may imply gradients of displacements as well. The contravariant components of displacements can be represented as,

$$U^j(X^k,\ t) \ = \ G^{ij}\ D_{j\alpha}(X^k)\ U_{\alpha N}(t) \tag{2.5-1a}$$

where G^{ij} is the contravariant metric tensors and $U_{\alpha N}$ are physical components of the generalised deformations at the nodes.

2. Displacement Gradients:

For convenience, the gradients of displacements can be approximated within element m by the corresponding generalized nodal gradients $R_{\beta N}(t)$ by,

$$U_{i|j} \ = \ H_{ij\beta}\ (X^k)\ R_{\beta N}(t) \tag{2.5-2}$$

$$\beta \ = \ 1,\ \ldots,\ \alpha_2$$

In equation (1.10-2), the constitutive expressions for $T^{(ji)}$ and $\hat{T}^{(kj)i}$ involve gradients of the displacements. These gradients are evaluated by differentiating Eq. (2.5-1). Thus they can be obtained in terms of the nodal displacements only. This means that the derivatives of the nodal displacements are not involved in the constitutive equations. However, (2.5-2) may be used for the displacement gradients in Eq. (1.10-2) in the last chapter. Hence $R_{\beta N}(t)$ is assumed to be independent variable and not function of the nodal displacements.

3. Stress Distribution:

The distribution of stresses can be similarly assumed to be,

$$T^{ij}\ (X^k,\ t) = S^{ij}{}_{\beta}\ (X^k)\ T_{\beta N}(t) \qquad (2.5\text{-}3)$$

$$\beta = 1,\ \ldots,\ \alpha_3$$

where $T_{\beta N}(t)$ are generalized nodal stresses and $S^{ij}{}_{\beta}$ are corresponding interpolation functions.

4. Temperature distribution:

In exactly similar way, the temperature distribution can be approximated in terms of generalized nodal temperatures $\theta_{\alpha N}$ as,

$$\theta(X^k,\ t) = \Phi_{\gamma}\ (X^k)\ \theta_{\gamma N}(t) \qquad (2.5\text{-}4)$$

$$\gamma = 1,\ldots,\ \alpha_4$$

The field variables $U_{\alpha N}$, $T_{\beta N}$ and $\theta_{\gamma N}$ need not be considered as tensor components, but the summation convention for the repeated indices has been used in the above expressions and will be retained throughout, unless otherwise stated.

2.6 Element Equations:

Introducing distributions (2.5-1) to (2.5-4) and applying summation extending over all the elements, Eq.(1.10-1) reduces to

$$\sum_{m=1}^{R} \dot{U}_{\alpha N}\ (C1_{\alpha\beta}\ T_{\beta N} - C2_{\alpha\alpha'}\ \ddot{U}_{\alpha' N} + L1_{\alpha}) = 0 \qquad (2.6\text{-}1)$$

where $\alpha, \alpha' = 1, \ldots, \alpha_1 ; \quad \beta = 1, \ldots, \alpha_3$

and $$C1_{\alpha\beta} = \int_{v_m} D_{i\alpha} S^{ji}{}_{\beta|j} \, dv - \int_{s_m} n_j D_{i\alpha} S^{ji}_{\beta} \, ds$$

$$C2_{\alpha\alpha'} = \int_{v_m} \rho_o D_{i\alpha} D^{i}{}_{\alpha'} \, dv \tag{2.6-2}$$

$$L1_{\alpha} = \int_{v_m} \rho_o D_{i\alpha} F^{i} dv + \int_{s_m} D_{i\alpha} P^{i} \, ds$$

The limit v_m corresponds to the volume occupied by the element m bounded by the surface s_m. In Eq. (2.6-1), $U_{\alpha N}$ will be common only to M surrounding elements among the total number R in the continuum. Again, since Eq. (2.6-1) remains true for any arbitrary $\dot{U}_{\alpha N}$ ($\neq 0$), it implies

$$\sum_{m=1}^{M}{}' \left(C1_{\alpha\beta} T_{\beta N} - C2_{\alpha\alpha'} \ddot{U}_{\alpha' N} + L1_{\alpha} \right) = 0 \tag{2.6-3}$$

where the summation Σ' extends over the surrounding M elements. Eq. (2.6-3) provides the governing equation corresponding to (1.10-1) in finite element model. Similarly, substitution of distribution (2.5-3) and (2.5-4) in (1.10-2) and (1.10-3) of Chapter 1, will result

$$\sum_{m=1}^{M}{}' \left(C3_{\beta\beta'} T_{\beta' N} - L2_{\beta} + L3_{\beta} \right) = 0 \tag{2.6-4}$$

and $$\sum_{m=1}^{M}{}' \left(C4_{\gamma\gamma'} \theta_{\gamma' N} - L4_{\gamma} + L5_{\gamma} + L6_{\gamma} \right) = 0 \tag{2.6-5}$$

where $\beta = 1, \ldots, \alpha_2$, $\beta' = 1, \ldots \alpha_3$; $\gamma, \gamma' = 1, \ldots, \alpha_4$

and
$$C3_{\beta\beta'} = \int_{v_m} H_{ij\beta} S^{ji}{}_{\beta'} \, dv$$

$$L2_\beta = \int_{v_m} H_{ij\beta} \left(\hat{T}^{(ji)} - \hat{T}^{(kj)i}{}_{|k}\right) dv - \int_{s_m} H_{ij\beta} n_k \hat{T}^{(kj)i} \, ds \qquad (2.6\text{-}6)$$

$$L3_\beta = \int_{v_m} \rho_o H_{ij\beta} F^{ji} \, dv + \int_{s_m} H_{ij\beta} P^{ji} \, ds$$

$$C4_{\gamma\gamma'} = \int_{v_m} \rho_o \Phi_{\gamma'} \frac{d}{dt} (\psi, \theta) \, dv$$

$$L4_\gamma = \int_{v_m} \Phi_{\gamma|i} Q^i \, dv - \int_{s_m} n_i \Phi_\gamma Q^i \, ds \qquad (2.6\text{-}7)$$

$$L5_\gamma = \int_{v_m} \rho_o q \Phi_\gamma \, dv$$

$$L6_\gamma = \int_{v_m} \Phi_\gamma \omega_\alpha \, dv$$

It may be noted that (2.6-3) provides three equations for each node, corresponding to displacement interpolation. The number of equations from (2.6-4) is six or nine per node of gradient interpolation, for the monopolar or dipolar cases respectively. Only one equation per node (for temperature interpolation) arises from (2.6-5).

In each element, the total number of equations from (2.6-3), (2.6-4) and (2.6-5) is $\alpha_1 + \alpha_2 + \alpha_4$, where as the

total number of unknowns are $\alpha_1 + \alpha_3 + \alpha_4$. Hence, for determinancy of solution, α_2 has to be equal to α_3.

The set of Eqs. (2.6-3) to (2.6-5) is nonlinear coupled simultaneous differential equations with respect to time t. Finite element transformation has reduced the continuous function of field variables, with respect to space coordinates, to a finite number of field unknowns rendering the system to be space independent. This reduces the complexity of the nonlinear equations and become more amenable for solution by computer operation at the expense of increased number of governing equations.

2.7 Expressions For Strains:

The above equations in the foregoing section are not complete without explicit functional relations of E_{ij} and E_{ijk}. It has been already mentioned that various expressions may be chosen for strains, their principal requirements being that they should be frame indifferent and may be treated as arguments of the internal energy expression, reflecting the physical behaviours of the material. For the sake of brevity, the strain measures given in (1.10-9) to (1.10-12) are presented here. For future usage, the following quantities have been expressed in terms of displacement distribution.

1. Infinitesimal strain:

$$\tilde{E}_{ij} = (U_{i|j} + U_{j|i})/2 = \frac{1}{2} \phi^{\alpha}{}_{(ij)} U_{\alpha N} \qquad (2.7\text{-}1)$$

2. Infinitesimal rotation:

$$\tilde{R}_{ij} = (U_{i|j} - U_{j|i})/2 = \frac{1}{2} \phi^{\alpha}{}_{|ij|} U_{\alpha N} \quad (2.7\text{-}2)$$

3. Classical Strain:

$$E_{ij} = \tilde{E}_{ij} + \frac{1}{2} G^{mn} (\tilde{E}_{mi} + \tilde{R}_{mi}) (\tilde{E}_{nj} + \tilde{R}_{nj})$$

$$= \frac{1}{2} (\phi^{\alpha}{}_{(ij)} + \phi^{\alpha\alpha'}{}_{(ij)} U_{\alpha' N}) U_{\alpha N} \quad (2.7\text{-}3)$$

where

$$\phi^{\alpha}{}_{(ij)} = D_{i\alpha|j} + D_{j\alpha|i} \quad (2.7\text{-}4)$$

$$\phi^{\alpha}{}_{|ij|} = D_{i\alpha|j} - D_{j\alpha|i} \quad (2.7\text{-}5)$$

$$\phi^{\alpha\alpha'}{}_{(ij)} = \frac{G^{mn}}{4} (\phi^{\alpha}{}_{(mi)} \phi^{\alpha'}{}_{(nj)} + \phi^{\alpha}{}_{(mi)} \phi^{\alpha'}{}_{|nj|}$$

$$+ \phi^{\alpha}{}_{|mi|} \phi^{\alpha'}{}_{(nj)} + \phi^{\alpha}{}_{|mi|} \phi^{\alpha'}{}_{|nj|}) \quad (2.7\text{-}6)$$

and G^{mn} is the metric tensor in undeformed state.

4. Dipolar Strain:

$$E_{ijk} = (\bar{\phi}^{\alpha}{}_{i(jk)} + \bar{\phi}^{(\alpha\alpha')}{}_{i(jk)} U_{\alpha' N}) U_{\alpha N} \quad (2.7\text{-}7)$$

where

$$\bar{\phi}^{\alpha}{}_{i(jk)} = D_{i\alpha|jk} \quad (2.7\text{-}8)$$

and $$\bar{\phi}^{(\alpha\alpha')}_{i(jk)} = G^{mn} D_{m\alpha|i} D_{n\alpha'|jk} \quad (2.7\text{-}9)$$

It is evident from (2.7-6) and (2.7-9) that if some nonlinear terms in E_{ij} and E_{ijk} are to be neglected (i.e., assuming them to be small in comparison to others due to some physical reasoning), they may be easily incorporated in the expressions

for $\phi^{(\alpha\beta)}_{(ij)}$ and $\bar{\phi}^{(\alpha\beta)}_{i(jk)}$. For the simplified expressions, care should be taken that they represent integrable functions[48].

5. Rivlin-Ericksen Tensors:

$$A_{ij} = \frac{1}{2} \phi^{\alpha}_{(ij)} \dot{U}_{\alpha N} \qquad (2.7\text{-}10)$$

$$A_{ijk} = \bar{\phi}^{\alpha}_{i(jk)} \dot{U}_{\alpha N} \qquad (2.7\text{-}11)$$

In all the expressions from (2.7-1) to (2.7-11), α and α' vary from 1 to α_1.

Although, the finite element equations are simpler than the original set, they are not so simple as to be amenable to straight forward solution, unless they represent a linear case. In the absence of any direct and definite procedure, the method of parametric differentiation technique has been employed to obtain a step-by-step solution.

2.8 Parametric Differentiation:

2.8.1 Introduction:

The solution field of a system may be regarded as a phase space, being functionally related with a forcing parameter causing the motion of the system from one phase point to another. The philosophy behind the technique of parametric differentiation is to evaluate at a certain phase point, the tangential stiffness of the system for its motion due to a small perturbation of the forcing parameter. According to this view point, the method may be described as a generalized form of either infinitesimal perturbation or incremental approach.

The technique of parametric differentiation was first discussed by Davidenko[49]. At a much later date, the characteristics of this method, was discussed by Yakolev[50]. Application of this method for the solution of different problems may be found from the works of Davidenko[51], Setlur and Kapoor[52], Jischke and Baron[53], Rubbert and Landahl[54]. The method essentially consists of choosing a parameter which may be assumed as an independent variable λ for all the field variables and for which the solution is known at some limiting initial value, say at $\lambda = \lambda_o$. The nonlinear equations with the boundary conditions, if any, are then differentiated with respect to λ to yield a system of linear equations with variable coefficients in terms of differentials of the field variables and may be solved employing any standard numerical integration technique. The solution of the problem can now be obtained by further integration of the differentials for an assumed step size at λ_o. The solution can thus be marched out from λ_o to any other value of λ. The method can be best demonstrated by taking a simple example. Say it is required to solve the nonlinear differential equation,

$$C\dot{U} + L = 0 \tag{A}$$

with boundary condition

$$U = U_{t_o} \quad \text{at} \quad t = t_o \tag{B}$$

where, $C = C(U, \dot{U})$, $L = L(t)$, $U = U(t)$ and $\dot{U} = dU/dt$

Choose the forcing parameter λ such that

$$\bar{L} = \bar{L}(t, \lambda) \text{ where } L(t) = \bar{L}(t, \lambda_1)$$

and
$$\bar{U} = \bar{U}(t, \lambda) \text{ where } U(t) = \bar{U}(t, \lambda_1)$$

Consequently, it is possible to construct

$$\bar{C} = \bar{C}(\bar{U}(t, \lambda), \dot{\bar{U}}(t, \lambda))$$

where,

$$C(t) = \bar{C}(\bar{U}(t, \lambda_1), \dot{\bar{U}}(t, \lambda_1))$$

Hence the new differential equation can be written as,

$$\bar{C} \cdot \dot{\bar{U}} + \bar{L} = 0 \qquad \text{(C)}$$

with boundary condition,

$$\bar{U}(t_o, \lambda) = U_{t_o} \qquad \text{(D)}$$

The function $\bar{U}$ has to be chosen in such a way that it satisfies (C) and (D) at $\lambda = \lambda_o$. Now differentiation of (C) and (D) with respect to λ yields,

$$\frac{\partial \bar{C}}{\partial \bar{U}} \bar{U}^* + \left(\frac{\partial \bar{C}}{\partial \dot{\bar{U}}} + \bar{C} \right) \dot{\bar{U}}^* + \bar{L}^* = 0 \qquad \text{(E)}$$

and

$$\bar{U}^*(t_o, \lambda) = 0 \qquad \text{(F)}$$

where,
$$\bar{U}^* = \frac{\partial \bar{U}}{\partial \lambda} \text{ and } \bar{L}^* = \frac{\partial \bar{L}}{\partial \lambda}$$

The modified equation (E) and (F) is linear with respect to $\bar{U}^*$ and with variable coefficients. It can be solved, at least numerically, and the solution will be $\bar{U}^*(t, \lambda_o)$. From this $\bar{U}(t, \lambda_o + \Delta\lambda)$ can be easily obtained by the integration,

$$\bar{U}(t, \lambda_o + \Delta\lambda) = \bar{U}(t, \lambda_o) + \int_{\lambda_o}^{\lambda_o+\Delta\lambda} \bar{U}^*(t, \lambda_o)\, d\lambda \qquad (G)$$

In similar way, the solution can be marched upto $\lambda = \lambda_1$ when $\bar{U}$ will be the same as U which is the solution of (A) and (B). The advantages of this method are that it allows the solution to be obtained throughout the range of λ and no iterative technique is needed.

For efficient execution of the quadrature in (G) the following two forward-integration methods are recommended:

1. Improved Euler-Cauchy method.
2. Runge-Kutta-Gill method.

Both the schemes are easily available in any text of numerical analysis.

2.8.2 Application to Continuum Equations:

For continuum mechanics problems, in general, there may be two types of perturbations in the forcing functions. Firstly, at a certain instant of time, there may be jump in the loading, the time taken for this jump being negligibly small. Secondly, if the evolution of the phase points is considered as a smooth path with respect to time, the perturbations may be applied corresponding to small increment of time. That is, in the second case perturbation parameter may be chosen as time, where as, in the first case it may be

chosen as incremental loading. However, for the sake of generality, the parameter will be designated as λ and all the field variables after finite element transformations, i.e., $U_{\alpha N}$, $T_{\beta N}$ and $\theta_{\gamma N}$ including the body and surface forces will be assumed to be implicit function of λ. Hence, the tangential stiffness will be related through equations which will be obtained by differentiating Eqs. (2.6-3) to (2.6-5) with respect to λ. It will be further assumed that all field variables are known at the starting point, $\lambda = \lambda_o$. The object is to march forward from λ_o upto a point defined as $\lambda = \lambda_1$, which is also close to λ_o. By successive repetitions, the solution can move to any desired value of λ corresponding to time or loading whatever may be the case. Hence, performing necessary differentiations on (2.6-3) to (2.6-5), the equation can be written in the form,

$$\sum_{m=1}^{M}{}' \left(C1_{\alpha\beta}\, T^{*}_{\beta N} - C2_{\alpha\alpha'}\, \ddot{U}^{*}_{\alpha' N} + L1^{*}_{\alpha} \right) = 0 \quad (2.8\text{-}1)$$

$$\sum_{m=1}^{M}{}' \left(C3_{\beta\beta'}\, T^{*}_{\beta' N} - L2^{*}_{\beta} + L3^{*}_{\beta} \right) = 0 \quad (2.8\text{-}2)$$

and

$$\sum_{m=1}^{M}{}' \left(C4_{\gamma\gamma'}\, \theta^{*}_{\gamma' N} + C4^{*}_{\gamma\gamma'}\, \theta_{\gamma' N} - L4^{*}_{\gamma} + L5^{*}_{\gamma} + L6^{*}_{\gamma} \right) = 0 \quad (2.8\text{-}3)$$

where $\alpha, \alpha' = 1, \ldots, \alpha_1$; $\beta, \beta' = 1, \ldots, \alpha_3$

and $\gamma\ \gamma' = 1, \ldots, \alpha_4$.

In Eqs. (2.8-1) to (2.8-3), the asterisked quantities are the derivatives with respect to λ of the corresponding unasterisked quantities. $T^*_{\beta N}$, $U^*_{\alpha N}$ and $\theta^*_{\gamma N}$ are respectively the derivatives of nodal stresses, displacements and temperatures. These are unknowns at this stage. $L1^*_\alpha$ and $L3^*_\beta$ are incremental mechanical loads and $L5^*_\gamma$ is increment of heat due to increase of q. These are known quantities. The other coefficients, i.e., $C4^*_{\gamma\gamma}$,, $L2^*_\beta$, $L4^*_\gamma$ and $L6^*_\gamma$ have to be expressed interms of the derivatives of the independent variables, $U^*_{\alpha N}$, $\theta^*_{\alpha N}$ and their time derivatives. These quantities are very complicated and since solution of any specific problem with this generality is out of scope of this work, no attempt will be made to obtain the explicit expressions of the above quantities. However, it is not very difficult to visualize that the expressions can be obtained and after substitution, Eqs. (2.8-1) to (2.8-3) will be linear differential equations with respect to time and in terms of $T^*_{\beta N}$, $U^*_{\alpha N}$ and $\theta^*_{\gamma N}$ which can be solved numerically. Once these quantities are obtained, the field variables, i.e., $T_{\beta N}$, $U_{\alpha N}$ and $\theta_{\gamma N}$ at $\lambda = \lambda_1$ can be easily evaluated through the quadratures,

$$U_{\alpha N} = U_{\alpha N}\Big|_{\lambda=\lambda_o} + \int_{\lambda_o}^{\lambda_1} U^*_{\alpha N}\, d\lambda$$

$$T_{\beta N} = T_{\beta N}\Big|_{\lambda=\lambda_o} + \int_{\lambda_o}^{\lambda_1} T^*_{\beta N}\, d\lambda \qquad (2.8\text{-}4)$$

and

$$\theta_{\gamma N} = \theta_{\gamma N}\Big|_{\lambda=\lambda_o} + \int_{\lambda o}^{\lambda_1} \theta^*_{\gamma N}\, d\lambda$$

The advantage here is that the dormant nonlinearity of the problem (since it is not quite apparent in (2.8-1) to (2.8-3)) is confined only to first order equations and has formally been isolated in the quadratures (2.8-4) only.

In the next chapter, application to a specific case concerning plane problems will be dealt with simplified assumptions on constitutive relations and explicit expressions of all the quantities will be obtained. This will further clearify the technique.

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CHAPTER 3

PLANE PROBLEM

3.1 Introduction:

Two dimensional plane problem is one of the simpler cases encountered in continuum mechanics, though it involves, in general, high degree of nonlinearity for geometry as well as for physical material properties. This fact has made it rather difficult to obtain solution to any but the simplest problem by traditional methods. However, with the advent of high speed computers and development of finite element technique, the problems can be attacked with comparative ease.

Geometrically and physically nonlinear problems particularly with elastoplastic material have been studied by Felippa[1] considering displacement models. Applications of this method to inelastic structures have been considered by Pope[2], Armen, Pifko and Levine[3], Yamada et al.[4], Isakson[5], Marcal and King[6]. Viscoelastic problems are investigated by Oden[7], White[8], Fried[9], while Oden and Krass[10], Funjino and Ohsaka[11] have studied thermal problems. Analysis of nonlinear geometrical problems are considered by Walker and Hall[12] and Oden[13].

In this Chapter, the finite element equations for a mixed rectangular model in cartesian coordinate system will

be developed for plane problems having nonlinear geometrical and physical properties associated with coupled thermal effect. In section 2, some simplifying assumptions have been made while section 2 and 3 have been devoted respectively for the development of the finite element equations and numerical solutions of some problems.

3.2 Simplification:

The basic equations which has been derived in Chapter 2 are quite general. Various practical restraints inhibit their use for the solution of engineering problems. For example, due to lack of experimental data, the properties of a material implied for dipolar stresses are not known. Moreover, the nonlinearities are bound to make the formulation quite complicated which needs justification for practical utility. Most severe restriction comes from the availability of computer time and the expenses to be incurred for satisfying academic inquisitiveness. Considering the above points, it seems justified to simplify the problem having the following restrictions.

1. The motion has been assumed to be quasistatic and any term involving acceleration will be neglected.

2. The dipolar stress field is nonexistent.

3. It is assumed that the resulting deformations are accompanied by large rotations. This implies that the squares

of the rotations are comparable to the strains. Thus nonlinearities are due to the squares of rotations in the strain-displacement relations.

4. Generally, the effect of deformation on thermal state is small. Hence the nonlinearity of strains in the thermal equation will be neglected.

5. The constitutive relation for heat flux vector shall be given by generalized form of Fourier law, such that,

$$Q^i = K^{ij}(\theta) \cdot \theta_{,j}$$

in cartesian coordinate system. Here, comma denotes the partial differentiation with respect to the subscript following it.

6. In the equation of thermal state, the variation of the constitutive coefficients (e.g., $(C^{(ij)(kl)})^* = (\partial\hat{T}^{(ij)}/\partial E_{kl})^*$) will be neglected.

Considering the above restrictions, the constitutive equations can be prescribed as follows,

$$\hat{T}^{(ji)} = \hat{\hat{T}}^{ji}(E_{kl}, A_{kl}, \theta, t^p)$$

$$_d\hat{T}^{(ji)} = {}_d\hat{\hat{T}}^{ji}(E_{kl}, A_{kl}, \theta, t^p) \qquad (3.2\text{-}1)$$

$$\psi = \hat{\hat{\psi}}(E_{kl}, A_{kl}, \theta, t^p)$$

and

$$Q^i = K^{ij}(\theta)\,\theta_{,j}$$

where,

$$E_{kl} = (U_{k,l}+U_{l,k})/2 + \frac{1}{8}(U_{m,k}-U_{k,m})(U_{m,l}-U_{l,m})$$

$$A_{kl} = (\dot{U}_{kl} + \dot{U}_{l,k})/2$$

In the above constitutive equations dependence of past history has been tacitly assumed through the argument t^p.

3.3 Finite Element Derivation:

For the construction of finite element models consider the plane continuum to be represented by R number of rectangular elements in cartesian coordinate system, with a typical element as m, a typical node as p and the number of nodes, M, surrounding a node p as shown in Fig. 3-1. The independent field variables are displacements U_i, stresses T^{ji} ($= T^{ij}$) and temperature θ, where i, j take values of 1, 2. Hence the number of degrees of freedom per node is six and dimension of the stiffness matrices per element will be 24 x 24. The distribution of each of the field variables has been assumed linear in X^1 and X^2 direction. Hence, interpolating function for a typical field variable, say Y, having nodal values as Y_1, Y_2, Y_3 and Y_4 can be expressed in the form,

$$Y = \langle O_1 \quad O_2 \quad O_3 \quad O_4 \rangle \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{Bmatrix} \qquad (3.3\text{-}1)$$

where,

$$\begin{aligned} O_1 &= (ab - bX^1 - bX^2 + X^1X^2)/ab \\ O_2 &= (bX^1 - X^1X^2)/ab \\ O_3 &= X^1X^2 / ab \\ \text{and} \quad O_4 &= (aX^2 - X^1X^2)/ab \end{aligned} \qquad (3.3\text{-}2)$$

where X^1 and X^2 are local coordinates and a, b are typical dimensions of an element (Fig. 3-2). The nodal displacements, stresses and temperatures have been arranged in the sequence,

$$\{U_{\alpha N}\} = \langle U_{11}\ U_{12}\ U_{21}\ U_{22}\ U_{31}\ U_{32}\ U_{41}\ U_{42}\rangle^T$$

$$\{T_{\beta N}\} = \langle T_1^{11}\ T_1^{12}\ T_1^{22}\ T_2^{11}\ T_2^{12}\ \ldots .T_4^{12}\ T_4^{22}\rangle^T \quad (3.3\text{-}3)$$

and $\{\theta_{\gamma N}\} = \langle\ \theta_1\ \ \theta_2\ \ \theta_3\ \ \theta_4\ \rangle^T$

where $U_{\delta i}$, T_δ^{ij} and θ_δ, i, j = 1,2; δ = 1, ..., 4 are nodal values of the corresponding quantities and i, j denotes coordinate directions in conventional way. Distributions of $H_{ij\beta}$ shall be taken same as those of T^{ij}. To obtain, explicitly, the matrices in the governing equations, the following steps are necessary.

1. Substituting the distributions (3.3-1) in $(2.6\text{-}2)_1$ and taking into account that the surface integral in $(2.6\text{-}2)_1$ shall have no effect on the assembled structure because of continuity of stresses and displacements, $C1_{\alpha\beta}$ can be obtained after integration and has been given in Table (3-1). Similarly, the matrix $C3_{\beta\beta'}$ has been shown in Table (3-2). The vector $L1_\alpha$ can be easily evaluated for different cases of body force and surface load distributions.

2. Evaluation of $L2^*_\beta$:

Simplifying the expression for $L2_\beta$ in (2.6-6), the incremental $L2^*_\beta$ can be written as,

$$L2^*_\beta = \int_{A_m} H_{ij\beta}\, \hat{T}^{(ji)*}\, dA \tag{3.3-4}$$

where A_m is the area of integration for the element m and

$$\hat{T}^{(ji)*} = \frac{1}{2}\, \hat{\hat{T}}^{(kl)*} \cdot \left(\frac{\partial E_{kl}}{\partial U_{i,j}} + \frac{\partial E_{kl}}{\partial U_{j,i}} \right)$$

$$+ \frac{1}{2}\, \hat{\hat{T}}^{(kl)} \cdot \left(\frac{\partial E_{kl}}{\partial U_{i,j}} + \frac{\partial E_{kl}}{\partial U_{j,i}} \right)^* \tag{3.3-5}$$

It can be shown that for the nonlinear strain expression $(3.2\text{-}1)_5$ chosen for this case,

$$\frac{\partial E_{kl}}{\partial U_{i,j}} + \frac{\partial E_{kl}}{\partial U_{j,i}} = 2\delta^k{}_j\, \delta^l{}_i \tag{3.3-6}$$

where $\delta^k{}_j$ is Kronecker delta. Hence $T^{(ji)*}$ can be written as,

$$\hat{T}^{(ji)*} = C^{(ji)(kl)}\, E^*_{kl} + \hat{C}^{(ji)(kl)} A^*_{kl} + \underset{\sim}{C}^{(ji)}\, \theta^* \tag{3.3-7}$$

where $\underset{\sim}{C}^{(ji)(kl)} = \dfrac{\partial \hat{\hat{T}}^{(ji)}}{\partial E_{kl}}$; $\tilde{C}^{(ji)(kl)} = \dfrac{\partial \hat{\hat{T}}^{(ji)}}{\partial A_{kl}}$

and $\underset{\sim}{C}^{(ji)} = \dfrac{\partial \hat{\hat{T}}^{(ji)}}{\partial \theta}$ (3.3-8)

Now after substituting the expression for $\hat{T}^{(ji)*}$ from (3.3-7) in terms of nodal field variables in Eq. (3.3-4), $L2^*_\beta$ can be written as,

$$L2^*_\beta = |I^1_{\beta\alpha}|\, \{U^*_{\alpha N}\} + |I^2_{\beta\alpha}|\, \{U^*_{\alpha N}\} + |\tilde{I}^1_{\beta\alpha}|\, \{\dot{U}^*_{\alpha N}\}$$

$$+ |\underset{\sim}{I}_{\beta\gamma}|\, \{\theta^*_{\gamma N}\} \tag{3.3-9}$$

where $|I^1_{\beta\alpha}|$, $|I^2_{\beta\alpha}|$ and $|\underset{\sim}{I}_{\beta\gamma}|$ are given in Tables 3-3 to

3-5 respectively. Matrix $\tilde{I}^{1}_{\beta\alpha}$ will be same as $I^{1}_{\beta\alpha}$ only the coefficients $C^{(ji)(kl)}$ have to be replaced by $\tilde{C}^{(ji)(kl)}$. For convenience, linear and nonlinear parts of the strain expressions have been separated in $|I^{1}_{\beta\alpha}|$ and $|I^{2}_{\beta\alpha}|$ respectively.

3. Expressions for $C4_{\gamma\gamma'}$ and $C4^{*}_{\gamma\gamma'}$.

The free energy expression has been chosen as,

$$\rho_o \ \psi = \rho_o \hat{\hat{\psi}} \ (E_{ij}, A_{ij}, \theta)$$

therefore,

$$\rho_o \frac{d\psi}{dt} \left(\frac{\partial\psi}{\partial\theta}\right) = C^{ij}_{\theta} \dot{E}_{ij} + \tilde{C}^{ij}_{\theta} \dot{A}_{ij} + C^{\theta}_{\theta} \dot{\theta} \qquad (3.3\text{-}10)$$

where $$C^{ij}_{\theta} = \rho_o \frac{\partial\psi}{\partial E_{ij}} \left(\frac{\partial}{\partial\theta}\right), \quad \tilde{C}^{ij}_{\theta} = \rho_o \frac{\partial}{\partial A_{ij}} \left(\frac{\partial\psi}{\partial\theta}\right) \qquad (3.3\text{-}11)$$

and $$C^{\theta}_{\theta} = \rho_o \frac{\partial}{\partial\theta} \left(\frac{\partial\psi}{\partial\theta}\right) \qquad (3.3\text{-}12)$$

Since the effect of deformations on thermal state is generally assumed to be small, the nonlinear part of E_{ij} may be neglected for thermal equation. Moreover, the motion has been assumed to be quasistatic, i.e., the effect of $\dot{A}_{ij}$ is not prominent relative to others. Hence substituting in $(2.6\text{-}7)_1$ the simplified expression,

$$\rho_o \frac{d}{dt} \left(\frac{\partial\psi}{\partial\theta}\right) = C^{ij}_{\theta} \dot{E}^{1}_{ij} + C^{\theta}_{\theta} \dot{\theta} \qquad (3.3\text{-}13)$$

where $\dot{E}^{1}_{ij}$ is the linear part of $\dot{E}_{ij}$ and performing necessary integration, $C4_{\gamma\gamma'}$ can be written as,

$$C4_{\gamma\gamma'} = |Y_{\gamma\gamma'}| + |\tilde{Y}_{\gamma\gamma'}| \qquad (3.3\text{-}14)$$

where,

$$|Y_{\gamma\gamma'}| = \begin{bmatrix} \bar{J}_1 & \bar{J}_2 & \bar{J}_3 & \bar{J}_4 \\ & \bar{J}_5 & \bar{J}_6 & \bar{J}_7 \\ \text{Symm.} & & \bar{J}_8 & \bar{J}_9 \\ & & & \bar{J}_{10} \end{bmatrix} \qquad (3.3\text{-}15)$$

$$|\underset{\sim}{Y}_{\gamma\gamma'}| = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 & \bar{K}_4 \\ & \bar{K}_5 & \bar{K}_6 & \vec{\bar{\bar{K}}}_7 \\ & & \bar{K}_8 & \bar{K}_9 \\ \text{Symm.} & & & \bar{K}_{10} \end{bmatrix} \qquad (3.3\text{-}16)$$

$$\bar{J}_p = < \dot{U}_{\alpha N} > \{ J_p \} \qquad (3.3\text{-}17)$$

$$\bar{K}_p = < \dot{\theta}_N > \{ K_p \} \;, \quad p = 1, 2, \ldots, 10 \qquad (3.3\text{-}18)$$

and matrices $\{ J_p \}$ and $\{ K_p \}$ have been given in Tables 3-7 and 3-8 respectively.

After some matrix manipulation, the incremental form of $C4_{\gamma\gamma'}$ can be given by,

$$C4^*_{\gamma\gamma'} \; \theta_{\gamma' N} = |Y^1_{\gamma\alpha}| \{ \dot{U}^*_{\alpha N} \} + | \underset{\sim}{Y}^1_{\gamma\gamma'} | \{ \dot{\theta}^*_{\gamma' N} \} \qquad (3.3\text{-}19)$$

where,

$$| Y^1_{\gamma\alpha'} | \quad \begin{bmatrix} J_1^T & J_2^T & J_3^T & J_4^T \\ & J_5^T & J_6^T & J_7^T \\ \text{Symm.} & & J_8^T & J_9^T \\ & & & J_{10}^T \end{bmatrix} \{ \theta_{\gamma N} \} \qquad (3.3\text{-}20)$$

$$|\underset{\sim}{Y}^{1}_{\gamma\gamma'}| = \begin{bmatrix} K_1^T & K_2^T & K_3^T & K_4^T \\ & K_5^T & K_6^T & K_7^T \\ \text{Symm.} & & K_8^T & K_9^T \\ & & & K_{10}^T \end{bmatrix} \{\theta_{\gamma N}\} \quad (3.3\text{-}21)$$

4. Expression for $L4^*_\gamma$:

Since the constitutive equation for heat flux vector has been accepted as,

$$Q^i = K^{ij}(\theta) \cdot \theta,_j \qquad (3.3\text{-}22)$$

the expression for $L4_\gamma$ can be written by substituting, proper distribution and integrating $(2.6\text{-}7)_2$. Thus,

$$L4_\gamma = |R_{\gamma\gamma'}|\{\theta_{\gamma' N}\} \qquad (3.3\text{-}23)$$

where the matrix $|R_{\gamma\gamma'}|$ has been given in Table (3-9). Assuming now that K^{ij} is approximately constant for a small variation of θ, the incremental form of $L4_\gamma$ can be written as,

$$L4^*_\gamma = |R_{\gamma\gamma'}|\{\theta^*_{\gamma' N}\} \qquad (3.3\text{-}24)$$

5. Expression for $L6^*$:

Noting,

$$w_d = \frac{1}{2}\ {}_d\hat{T}^{ij}\,(\dot{U}_{i,j} + \dot{U}_{j,i}) \qquad (3.3\text{-}25)$$

the expression for $L6_\gamma$ can be written from $(3.6\text{-}7)_4$ in the form,

$$L6_\gamma = |S_{\gamma\alpha}|\{\dot{J}_{\alpha N}\} \qquad (3.3\text{-}26)$$

where $|S_{\gamma\alpha}|$ has been given in Table(3-10). Again assuming that the contribution of ${}_d\hat{T}^{(ij)*}$ will be negligible in the thermal equation, $L6^*_\gamma$ can be written as,

$$L6^*_\gamma = |S_{\gamma\alpha}|\{\dot{U}^*_{\alpha N}\} \tag{3.3-27}$$

Substitution of the above expressions in Eqs.(2.8-1) to (2.8-3) yields the following incremental forms of the finite element equations:

$$\sum_{m=1}^{M}{}' \left(|C1_{\alpha\beta}|\{T^*_{\beta N}\} - \{L1^*_\alpha\} \right) = 0 \tag{3.3-28}$$

$$\sum_{m=1}^{M}{}' \{ |C^3_{\beta\beta'}|\{T^*_{\beta',N}\} - (|I^1_{\beta\alpha}| + |I^2_{\beta\alpha}|)\{U^*_{\alpha N}\} - |\tilde{I}^1_{\beta\alpha}|\{\dot{U}^*_{\alpha N}\} - |\underset{\sim}{I}_{\beta N}|\{\theta^*_{\gamma N}\}\} = 0 \tag{3.3-29}$$

and

$$\sum_{m=1}^{M}{}' \{ |C4_{\gamma\gamma'}| - |R_{\gamma\gamma'}|\{\theta^*_{\gamma',N}\} + |\underset{\sim}{Y}^1_{\gamma\gamma'}|\{\dot{\theta}^*_{\gamma',N}\} + (|Y^1_{\gamma\alpha}| + |S_{\gamma\alpha}|)\{\dot{U}^*_{\alpha N}\} = 0 \tag{3.3-30}$$

These equations can be solved by the method outlined in Chapter 2.

$$\frac{1}{12}\begin{bmatrix}
-2b\;-2a & & 2b\;-a & & b\;\;a & & -b\;\;2a & \\
 & -2b\;-2a & & 2b\;-a & & b\;\;a & & -b\;\;2a \\
-2b\;-a & & 2b\;-2a & & b\;\;2a & & -b\;\;a & \\
 & -2b\;-a & & 2b\;-2a & & b\;\;2a & & -b\;\;a \\
-b\;-a & & b\;-2a & & 2b\;\;2a & & -2b\;\;a & \\
 & -b\;-a & & b\;-2a & & 2b\;\;2a & & -2b\;\;a \\
-b\;-2a & & b\;-a & & 2b\;\;a & & -2b\;\;2a & \\
 & -b\;-2a & & b\;-a & & 2b\;\;a & & -2b\;\;2a
\end{bmatrix}$$

Table 3-1: Matrix $C1_{\alpha\beta}$.

$$\frac{ab}{36}\begin{bmatrix}
4 & & & 2 & & & 1 & & & 2 & & \\
 & 4 & & & 2 & & & 1 & & & 2 & \\
 & & 4 & & & 2 & & & 1 & & & 2 \\
2 & & & 4 & & & 2 & & & 1 & & \\
 & 2 & & & 4 & & & 2 & & & 1 & \\
 & & 2 & & & 4 & & & 2 & & & 1 \\
1 & & & 2 & & & 4 & & & 2 & & \\
 & 1 & & & 2 & & & 4 & & & 2 & \\
 & & 1 & & & 2 & & & 4 & & & 2 \\
2 & & & 1 & & & 2 & & & 4 & & \\
 & 2 & & & 1 & & & 2 & & & 4 & \\
 & & 2 & & & 1 & & & 2 & & & 4
\end{bmatrix}$$

Table 3-2: Matrix $C3_{\beta\beta}$.

$$
12\left[\begin{matrix}
-2bc^{11}-2ac^{13} & -2ac^{12}-2bc^{13} & 2bc^{11}-ac^{13} & 2ac^{12}-bc^{13}\\
-2bc^{31}-2ac^{33} & -2ac^{32}-2bc^{33} & 2bc^{31}-ac^{33} & 2ac^{32}-bc^{33}\\
-2bc^{21}-2ac^{23} & -2ac^{22}-2bc^{23} & 2bc^{21}-ac^{23} & 2ac^{22}-bc^{23}\\
-2bc^{11}-ac^{13} & -2ac^{12}-bc^{13} & 2bc^{11}-2ac^{13} & 2ac^{12}-2bc^{13}\\
-2bc^{31}-ac^{33} & -2ac^{32}-bc^{33} & 2bc^{31}-2ac^{33} & 2ac^{32}-2bc^{33}\\
-2bc^{21}-ac^{23} & -2ac^{22}-bc^{23} & 2bc^{21}-2ac^{23} & 2ac^{22}-2bc^{23}\\
-bc^{11}-ac^{13} & -ac^{12}-bc^{13} & bc^{11}-2ac^{13} & ac^{12}-2bc^{13}\\
-bc^{31}-ac^{33} & -ac^{32}-bc^{33} & bc^{31}-2ac^{33} & ac^{32}-2bc^{33}\\
-bc^{21}-ac^{23} & -ac^{22}-bc^{23} & bc^{21}-2ac^{23} & ac^{22}-2bc^{23}\\
-bc^{11}-2ac^{13} & -ac^{12}-2bc^{13} & bc^{11}-ac^{13} & ac^{12}-bc^{13}\\
-bc^{31}-2ac^{33} & -ac^{32}-2bc^{33} & bc^{31}-ac^{33} & ac^{32}-bc^{33}\\
-bc^{21}-2ac^{23} & -ac^{22}-2bc^{23} & bc^{21}-ac^{23} & ac^{22}-bc^{23}
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
bc^{11}+ac^{13} & ac^{12}+bc^{13} & -bc^{11}+2ac^{13} & -ac^{12}+2bc^{13}\\
bc^{31}+ac^{33} & ac^{32}+bc^{33} & -bc^{31}+2ac^{33} & -ac^{32}+2bc^{33}\\
bc^{21}+ac^{23} & ac^{22}+bc^{23} & -bc^{21}+2ac^{23} & -ac^{22}+2bc^{23}\\
bc^{11}+2ac^{13} & ac^{12}+2bc^{13} & -bc^{11}+ac^{13} & -ac^{12}+bc^{13}\\
bc^{31}+2ac^{33} & ac^{32}+2bc^{33} & -bc^{31}+ac^{33} & -ac^{32}+bc^{33}\\
bc^{21}+2ac^{23} & ac^{22}+2bc^{23} & -bc^{21}+ac^{23} & -ac^{22}+bc^{23}\\
2bc^{11}+2ac^{13} & 2ac^{12}+2bc^{13} & -2bc^{11}+ac^{13} & -2ac^{12}+bc^{13}\\
2bc^{31}+2ac^{33} & 2ac^{32}+2bc^{33} & -2bc^{31}+ac^{33} & -2ac^{32}+bc^{33}\\
2bc^{21}+2ac^{23} & 2ac^{22}+2bc^{23} & -2bc^{21}+ac^{23} & -2ac^{22}+bc^{23}\\
2bc^{11}+ac^{13} & 2ac^{12}+bc^{13} & -2bc^{11}+2ac^{13} & -2ac^{12}+2bc^{13}\\
2bc^{31}+ac^{33} & 2ac^{32}+bc^{33} & -2bc^{31}+2ac^{33} & -2ac^{32}+2bc^{33}\\
2bc^{21}+ac^{23} & 2ac^{22}+bc^{23} & -2bc^{21}+2ac^{23} & -2ac^{22}+2bc^{23}
\end{matrix}\right]
$$

Notations:

$$
\begin{aligned}
&C^{(11)(11)} = C^{11};\quad C^{(12)(11)} = C^{31};\quad C^{(22)(11)} = C^{21}\\
&C^{(11)(12)} = C^{13};\quad C^{(12)(12)} = C^{33};\quad C^{(22)(12)} = C^{23}\\
&C^{(11)(22)} = C^{12};\quad C^{(12)(22)} = C^{32};\quad C^{(22)(22)} = C^{22}
\end{aligned}
$$

TABLE 3-3: Matrix $I^{1}{}_{\beta\alpha}$

$$
\begin{bmatrix}
d^1g^1 & d^1g^3 & d^1g^2 & -d^1g^3 & -d^1g^2 & d^1g^4 & -d^1g^1 & -d^1g^4 \\
d^2g^1 & d^2g^3 & d^2g^2 & -d^2g^3 & -d^2g^2 & d^2g^4 & -d^2g^1 & -d^2g^4 \\
d^3g^1 & d^3g^3 & d^3g^2 & -d^3g^3 & -d^3g^3 & d^3g^4 & -d^3g^1 & -d^3g^4 \\
d^1g^5 & d^1g^7 & d^1g^6 & -d^1g^7 & -d^1g^6 & d^1g^8 & -d^1g^5 & -d^1g^8 \\
d^2g^5 & d^2g^7 & d^2g^6 & -d^2g^7 & -d^2g^6 & d^2g^8 & -d^2g^5 & -d^2g^8 \\
d^3g^5 & d^3g^7 & d^3g^6 & -d^3g^7 & -d^3g^6 & d^3g^8 & -d^3g^5 & -d^3g^8 \\
d^1g^9 & d^1g^{11} & d^1g^{10} & -d^1g^{11} & -d^1g^{10} & d^1g^{12} & -d^1g^9 & -d^1g^{12} \\
d^2g^9 & d^2g^{11} & d^2g^{10} & -d^2g^{11} & -d^2g^{10} & d^2g^{12} & -d^2g^9 & -d^2g^{12} \\
d^3g^9 & d^3g^{11} & d^3g^{10} & -d^3g^{11} & -d^3g^{10} & d^3g^{12} & -d^3g^9 & -d^3g^{12} \\
d^1g^{13} & d^1g^{15} & d^1g^{14} & -d^1g^{15} & -d^1g^{14} & d^1g^{16} & -d^1g^{13} & -d^1g^{16} \\
d^2g^{13} & d^2g^{15} & d^2g^{14} & -d^2g^{15} & -d^2g^{14} & d^2g^{16} & -d^2g^{13} & -d^2g^{16} \\
d^3g^{13} & d^3g^{15} & d^3g^{14} & -d^3g^{15} & -d^3g^{14} & d^3g^{16} & -d^3g^{13} & -d^3g^{16}
\end{bmatrix}
$$

Notations:

$$d^1 = C^{(11)(11)} + C^{(11)(22)}$$

$$d^2 = C^{(12)(11)} + C^{(12)(22)}$$

$$d^3 = C^{(22)(11)} + C^{(22)(22)}$$

$$g^p = \frac{1}{288} \langle U_{\alpha N} \rangle \{ \tilde{g}^p \}$$

where $\{ \tilde{g}^p \}$ has been defined in Table 3-6 .

TABLE 3-4: Matrix $I^2_{\beta\alpha}$.

$$\frac{ab}{36}\begin{bmatrix}
4\tilde{C}^{11} & 2\tilde{C}^{11} & \tilde{C}^{11} & 2\tilde{C}^{11} \\
4\tilde{C}^{12} & 2\tilde{C}^{12} & \tilde{C}^{12} & 2\tilde{C}^{12} \\
4\tilde{C}^{22} & 2\tilde{C}^{22} & \tilde{C}^{22} & 2\tilde{C}^{22} \\
2\tilde{C}^{11} & 4\tilde{C}^{11} & 2\tilde{C}^{11} & \tilde{C}^{11} \\
2\tilde{C}^{12} & 4\tilde{C}^{12} & 2\tilde{C}^{12} & \tilde{C}^{12} \\
2\tilde{C}^{22} & 4\tilde{C}^{22} & 2\tilde{C}^{22} & \tilde{C}^{22} \\
\tilde{C}^{11} & 2\tilde{C}^{11} & 4\tilde{C}^{11} & 2\tilde{C}^{11} \\
\tilde{C}^{12} & 2\tilde{C}^{12} & 4\tilde{C}^{12} & 2\tilde{C}^{12} \\
\tilde{C}^{22} & 2\tilde{C}^{22} & 4\tilde{C}^{22} & 2\tilde{C}^{22} \\
2\tilde{C}^{11} & \tilde{C}^{11} & 2\tilde{C}^{11} & 4\tilde{C}^{11} \\
2\tilde{C}^{12} & \tilde{C}^{12} & 2\tilde{C}^{12} & 4\tilde{C}^{12} \\
2\tilde{C}^{22} & \tilde{C}^{22} & 2\tilde{C}^{22} & 4\tilde{C}^{22}
\end{bmatrix}.$$

Table 3-5: Matrix $\tilde{I}_{\beta\gamma}$.

$$\tilde{g}^1 = \begin{Bmatrix} 9a/b \\ -8 \\ 3a/b \\ 8 \\ -3a/b \\ 4 \\ -9a/b \\ -4 \end{Bmatrix} \quad \tilde{g}^2 = \begin{Bmatrix} 3a/b \\ -4 \\ 3a/b \\ 4 \\ -3a/b \\ 2 \\ -3a/b \\ -2 \end{Bmatrix} \quad \tilde{g}^3 = \begin{Bmatrix} -8 \\ 9b/a \\ -4 \\ -9b/a \\ 4 \\ -3b/a \\ 8 \\ 3b/a \end{Bmatrix} \quad \tilde{g}^4 = \begin{Bmatrix} 4 \\ -3b/a \\ 2 \\ 3b/a \\ -2 \\ 3b/a \\ -4 \\ -3b/a \end{Bmatrix}$$

Contd...

$$\tilde{g}^5 = \begin{Bmatrix} 3a/b \\ -4 \\ 3a/b \\ 4 \\ -3a/b \\ 2 \\ -3a/b \\ -2 \end{Bmatrix} \quad \tilde{g}^6 = \begin{Bmatrix} 3a/b \\ -8 \\ 9a/b \\ 8 \\ -9a/b \\ 4 \\ -3a/b \\ -4 \end{Bmatrix} \quad \tilde{g}^7 = \begin{Bmatrix} -4 \\ 9b/a \\ -8 \\ -9b/a \\ 8 \\ -3b/a \\ 4 \\ 3b/a \end{Bmatrix} \quad \tilde{g}^8 = \begin{Bmatrix} 2 \\ -3b/a \\ 4 \\ 3b/a \\ -4 \\ 3b/a \\ -2 \\ -3b/a \end{Bmatrix}$$

$$\tilde{g}^9 = \begin{Bmatrix} 3a/b \\ -2 \\ 3a/b \\ 2 \\ -3a/b \\ 4 \\ -3a/b \\ -4 \end{Bmatrix} \quad \tilde{g}^{10} = \begin{Bmatrix} 3a/b \\ -4 \\ 9a/b \\ 4 \\ -9a/b \\ 8 \\ -3a/b \\ -8 \end{Bmatrix} \quad \tilde{g}^{11} = \begin{Bmatrix} -2 \\ 3b/a \\ -4 \\ -3b/a \\ 4 \\ -3b/a \\ 2 \\ 3b/a \end{Bmatrix} \quad \tilde{g}^{12} = \begin{Bmatrix} 4 \\ -3b/a \\ 8 \\ 3b/a \\ -8 \\ 9b/a \\ -4 \\ -9b/a \end{Bmatrix}$$

$$\tilde{g}^{13} = \begin{Bmatrix} 9a/b \\ -4 \\ 3a/b \\ 4 \\ -3a/b \\ 8 \\ -9a/b \\ -8 \end{Bmatrix} \quad \tilde{g}^{14} = \begin{Bmatrix} 3a/b \\ -2 \\ 3a/b \\ 2 \\ -3a/b \\ 4 \\ -3a/b \\ -4 \end{Bmatrix} \quad \tilde{g}^{15} = \begin{Bmatrix} -4 \\ 3b/a \\ -2 \\ -3b/a \\ 2 \\ -3b/a \\ 4 \\ 3b/a \end{Bmatrix} \quad \tilde{g}^{16} = \begin{Bmatrix} 8 \\ -3b/a \\ 4 \\ 3b/a \\ -4 \\ 9b/a \\ -8 \\ -9b/a \end{Bmatrix}$$

Table 3-6: Matrices $\tilde{g}^p$ ($p = 1, \ldots, 16$)

$$J_1 = \frac{1}{72}\begin{Bmatrix} -6bc^1-6ac^3 \\ -6bc^3-6ac^2 \\ 6bc^1-2ac^3 \\ 6bc^3-2ac^2 \\ 2bc^1+2ac^3 \\ 2bc^3+2ac^2 \\ -2bc^1+6ac^3 \\ -2bc^3+6ac^2 \end{Bmatrix} \quad J_2 = \frac{1}{72}\begin{Bmatrix} -3bc^1-2ac^3 \\ -3bc^3-2ac^2 \\ 3bc^1-2ac^3 \\ 3bc^3-2ac^2 \\ bc^1+2ac^3 \\ bc^3+2ac^2 \\ -bc^1+2ac^3 \\ -bc^3+2ac^2 \end{Bmatrix} \quad J_3 = \frac{1}{72}\begin{Bmatrix} -bc^1-ac^3 \\ -bc^3-ac^2 \\ bc^1-ac^3 \\ bc^3-ac^2 \\ bc^1+ac^3 \\ bc^3+ac^2 \\ -bc^1+ac^3 \\ -bc^3+ac^2 \end{Bmatrix}$$

$$J_4 = \frac{1}{72}\begin{Bmatrix} -2bc^1-3ac^3 \\ -2bc^3-3ac^2 \\ 2bc^1-ac^3 \\ 2bc^3-ac^2 \\ 2bc^1+ac^3 \\ 2bc^3+ac^2 \\ -2bc^1+3ac^3 \\ -2bc^3+3ac^2 \end{Bmatrix} \quad J_5 = \frac{1}{72}\begin{Bmatrix} -6bc^1-2ac^3 \\ -6bc^3-2ac^2 \\ 6bc^1-6ac^3 \\ 6bc^3-6ac^2 \\ 2bc^1+6ac^3 \\ 2bc^3+6ac^2 \\ -2bc^1+2ac^3 \\ -2bc^3+2ac^2 \end{Bmatrix} \quad J_6 = \frac{1}{72}\begin{Bmatrix} -2bc^1-ac^3 \\ -2bc^3-ac^2 \\ 2bc^1-ac^3 \\ 2bc^3-3ac^2 \\ 2bc^1+3ac^3 \\ 2bc^3+3ac^2 \\ -2bc^1+ac^3 \\ -2bc^3+ac^2 \end{Bmatrix}$$

$$J_7 = \frac{1}{72}\begin{Bmatrix} -bc^1 - ac^3 \\ -bc^3 - ac^2 \\ bc^1 - ac^3 \\ bc^3 - ac^2 \\ bc^1 + ac^3 \\ bc^3 + ac^2 \\ -bc^1 + ac^3 \\ -bc^3 + ac^2 \end{Bmatrix} \qquad J_8 = \frac{1}{72}\begin{Bmatrix} -2bc^1 - 2ac^3 \\ -2bc^3 - 2ac^2 \\ 2bc^1 - 6ac^3 \\ 2bc^3 - 6ac^2 \\ 6bc^1 + 6ac^3 \\ 6bc^3 + 6ac^2 \\ -6bc^1 + 2ac^3 \\ -6bc^3 + 2ac^2 \end{Bmatrix}$$

$$J_9 = \frac{1}{72}\begin{Bmatrix} -bc^1 - 2ac^3 \\ -bc^3 - 2ac^2 \\ bc^1 - 2ac^3 \\ bc^3 - 2ac^2 \\ 3bc^1 + 2ac^3 \\ 3bc^3 + 2ac^2 \\ -3bc^1 + 2ac^3 \\ -3bc^3 + 2ac^2 \end{Bmatrix} \qquad J_{10} = \frac{1}{72}\begin{Bmatrix} -2bc^1 - 6ac^3 \\ -2bc^3 - 6ac^2 \\ 2bc^1 - 2ac^3 \\ 2bc^3 - 2ac^2 \\ 6bc^1 + 2ac^3 \\ 6bc^3 + 2ac^2 \\ -6bc^1 + 6ac^3 \\ -6bc^3 + 6ac^2 \end{Bmatrix}$$

Notations: $c^1 = c_\theta^{11}$; $c^2 = c_\theta^{22}$; $c^3 = c_\theta^{12}$

Table 3-7: Matrices J_p $(p = 1, \ldots, 10)$.

$$K_1 = \frac{ab}{144}\begin{Bmatrix} 3 \\ 1 \\ 3 \end{Bmatrix} \quad K_2 = \frac{ab}{144}\begin{Bmatrix} \\ 1 \\ 1 \end{Bmatrix} \quad K_3 = \frac{ab}{144}\begin{Bmatrix} \\ 1 \\ 1 \end{Bmatrix} \quad K_4 = \frac{ab}{144}\begin{Bmatrix} \\ 1 \\ 3 \end{Bmatrix} \quad K_5 = \frac{\quad}{144}\begin{Bmatrix} \\ 3 \\ 1 \end{Bmatrix}$$

$$K_6 = \frac{ab}{144}\begin{Bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{Bmatrix} \quad K_7 = \frac{ab}{144}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \quad K_8 = \frac{ab}{144}\begin{Bmatrix} 1 \\ 3 \\ 9 \\ 3 \end{Bmatrix} \quad K_9 = \frac{ab}{144}\begin{Bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{Bmatrix} \quad K_{10} = \frac{ab}{144}\begin{Bmatrix} 3 \\ 1 \\ 3 \\ 9 \end{Bmatrix}$$

Table 3-8: Matrices K_p $(p = 1, 2, \ldots, 10)$.

$$\frac{1}{6ab}\begin{bmatrix} \begin{matrix} 4b^2K^{11}+3abK^{12} \\ -3abK^{12} \\ -3abK^{12}-2a^2K^{22} \\ 2b^2K^{11}+3abK^{12}-4a^2K^{22} \end{matrix} & \begin{matrix} -4b^2K^{11}+3abK^{12} \\ +3abK^{12} \\ +3abK^{12}-4a^2K^{22} \\ -2b^2K^{11}+3abK^{12}-2a^2K^{22} \end{matrix} & \begin{matrix} -2b^2K^{11}-3abK^{12} \\ -3abK^{12} \\ +3abK^{12}+4a^2K^{22} \\ -4b^2K^{11}+3abK^{12}+2a^2K^{22} \end{matrix} & \begin{matrix} 2b^2K^{11}-3abK^{12} \\ +3abK^{12} \\ -3abK^{12}+2a^2K^{22} \\ 4b^2K^{11}-9abK^{12}+4a^2K^{22} \end{matrix} \end{bmatrix}$$

Table 3-9: Matrix $R_{\gamma\gamma}$'.

$$
12 \begin{bmatrix}
-2bf^1-2af^3 & -2af^2-2bf^3 & 2bf^1-af^3 & 2af^2-bf^3 & bf^1+af^3 & af^2+bf^3 & -bf^1+2af^3 & -af^2+2bf^3 \\
-2bf^1-af^3 & -2af^2-bf^3 & 2bf^1-2af^3 & 2af^2-2bf^3 & bf^1+2af^3 & af^2+2bf^3 & -bf^1+af^3 & -af^2+bf^3 \\
-bf^1-af^3 & -af^2-bf^3 & bf^1-2af^3 & af^2-2bf^3 & 2bf^1+2af^3 & 2af^2+2bf^3 & -2bf^1+af^3 & -2af^2+bf^3 \\
-bf^1-2af^3 & -af^2-2bf^3 & bf^1-af^3 & af^2-bf^3 & 2bf^1+af^3 & 2af^2+bf^3 & -2bf^1+2af^3 & -2af^2+2bf^3
\end{bmatrix}
$$

Notations: $f^1 = {}_dT^{(11)}$; $f^2 = {}_dT^{(22)}$; $f^3 = {}_dT^{(12)}$

Table 3-10: Matrix $S_{\gamma\alpha}$.

3.4 Numerical Examples and Comments:

3.4.1: To show the applicability of the finite element model, three problems have been solved for which the typical arrangements of elements are shown in Fig. (3-2). Apart from these examples, various other simple problems whose results are known had been solved to check the reliability of the program. The errors were within acceptable limit.

Example 1: The classical problem for an anisotropic half space with a concentrated load is studied and the stress distribution is shown in Fig. (3-3). The material for the half space is chosen as Topaz and the linear material coefficients have been given by Hearmon[14]. The modified coefficients[15] for the two dimensional case are,

$$C^{(11)(11)} = 4.24 \; ; \qquad C^{(11)(22)} = C^{(22)(11)} = 1.23$$
$$C^{(12)(12)} = 7.64 \; ; \qquad C^{(22)(22)} = 3.42$$

The accuracy of the solution has been compared through statical checks and the deviations have been found to be not more than 4 to 5 percent. Moreover, on the line of symmetry, the finite element solution compares satisfactorily with that given by Lekhnitskii[15].

Example 2: A viscoelastic cantilever beam has been analysed assuming the constitutive equations to be,

$$(T^{11}, T^{22}) = (E_{11}, E_{22}) + 0.5\,(\dot{E}_{11}, \dot{E}_{22})$$
$$T^{12} = 0.5\,E_{12} + 0.25\,\dot{E}_{12}$$

This problem has been treated both for linear as well as nonlinear geometry and the maximum deflections at the free end and the deflection along the beam at time t = 1.5 have been shown in Fig. (3-4). The stresses, obtained from the solution, have excellent agreement with the theoretical values. But, since they do not have any special significance for viscoelasticity (determinate problem) they have not been drawn here.

Example 3: Second Danilovskaya problem has been selected to show the accuracy of the model for thermal analysis. The finite element solution and exact values as shown by Nickell and Sackman[17], have been compared in Fig. (3-5). For the comparison the following values are assumed,

$$\rho_o = 0.008 , \qquad \alpha = 15.3 \times 10^{-6}$$
$$\rho_o C_v = 0.96 \qquad K = 0.04$$
$$\lambda = 42.0 \times 10^4 \qquad \mu = 4.0 \times 10^4$$

where ρ_o, α, C_v , K, λ and μ are respectively mass density, linear thermal expansion coefficient, specific heat per unit mass, thermal conductivity and isothermal Lame's coefficients.

The convective heat transfer across the surface is assumed to follow the relation,

$$Q^i = h(T - T_o)$$

where direction i is normal to the boundary surface, h is the film constant which is assumed as 0.78, T is the temperature

of the surface and T_0 which is the ambient outside temperature, is taken as 300. The boundary surface temperature is prescribed to vary linearly from 300 at $t = 0$ to 600 at $t = 1.45 \times 10^{-10}$ and then it is kept constant.

From the Fig. (3-5), it is seen that U_2 and T agree satisfactorily upto $t = 1.60 \times 10^{-10}$. After that they start deviating rather sharply. But even before $t = 1.45 \times 10^{-10}$ the accuracy for stress is rather poor, the maximum error being about 8.0 percent. Unfortunately, due to limitation of Computer time, no further refinement for time increment is possible to demonstrate the error characteristics.

3.4.2 Comments:

From a few problems solved in this chapter, it is not possible to give any general idea regarding the error propagation for the particular model chosen here. However, it has been found that better results can be achieved by decreasing the element size and step length at the expense of execution time. This has particularly been verified for the solution having short range duration. But for long range solutions, by decreasing the step lengths, the results come nearer to the actual values, but it has not been possible to take the step length so small that the propagation error is negligible even after an appreciable lapse of time.

Moreover, during the solutions of various small problems for checking the program, it has been found that fixed end boundary conditions yield less accurate results than a free or simple supported end. This is possibly because of linear interpolations adopted for the model.

Continuum problems are, as such, quite complex and their detailed derivations are extremely complicated and massive. For this reason, the higher order interpolation laws become virtually prohibitive and it becomes mandatory to use linear law in spite of its disadvantages. However, the author feels that the results can be substantially improved, at least near the boundary zone, if some interpolation law can be used to impose such that the adjacent nodes can be made to follow certain law so that the boundary conditions can be satisfied more accurately.

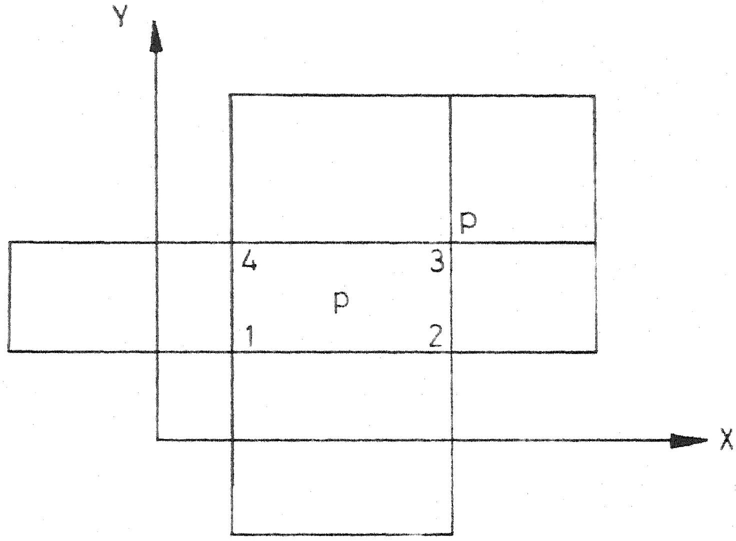


FIG. (3-1) ARRANGEMENT OF ELEMENTS

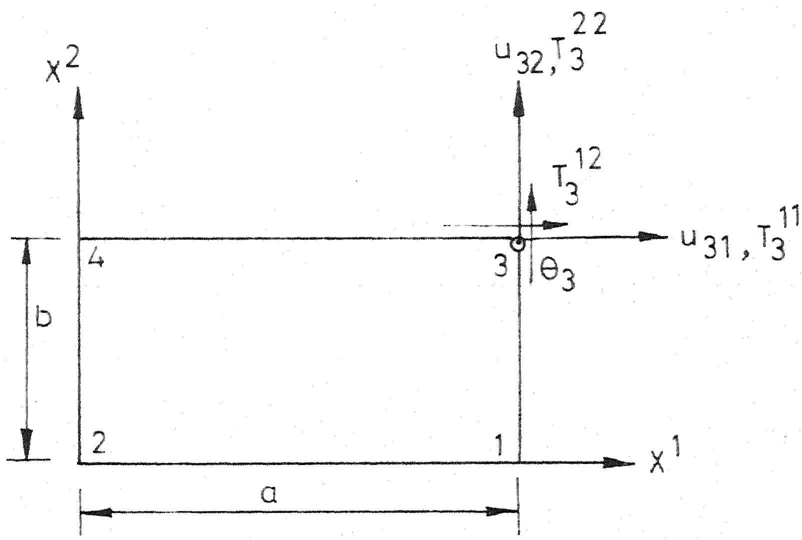
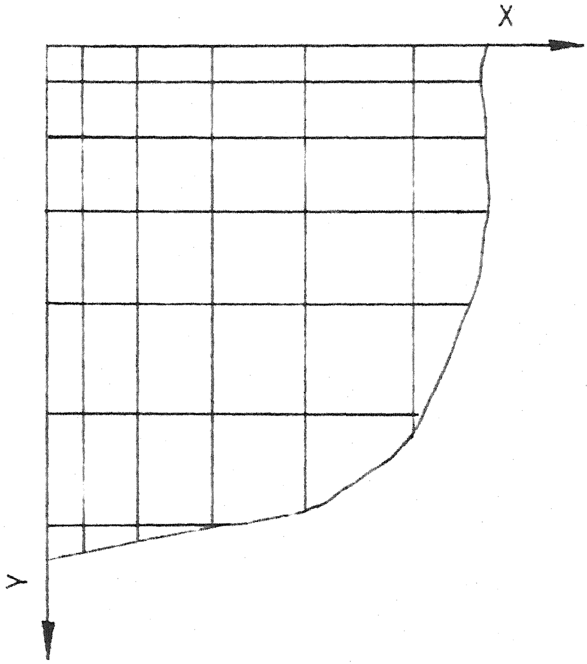
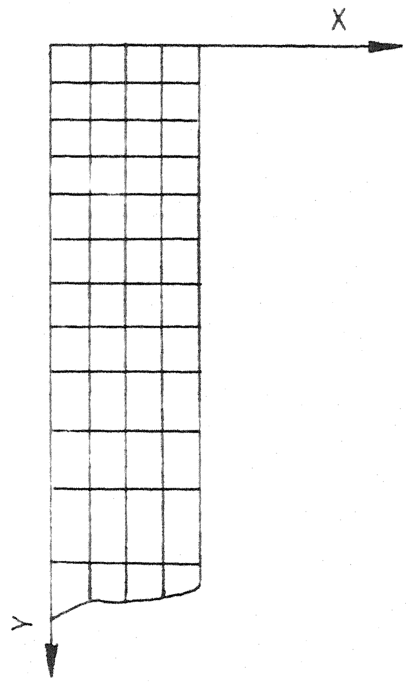


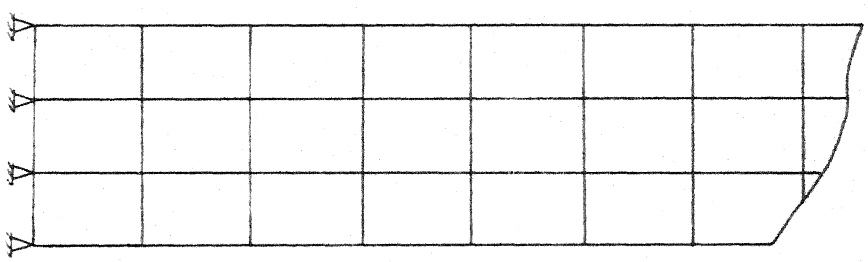
FIG. (3-2) LOCAL COORDINATES AND FIELD VARIABLES



) Half space for conc. load



(2) Half space for thermal analysis



(3) Viscoelastic beam

FIG. (3-3) ARRANGEMENT OF ELEMENTS

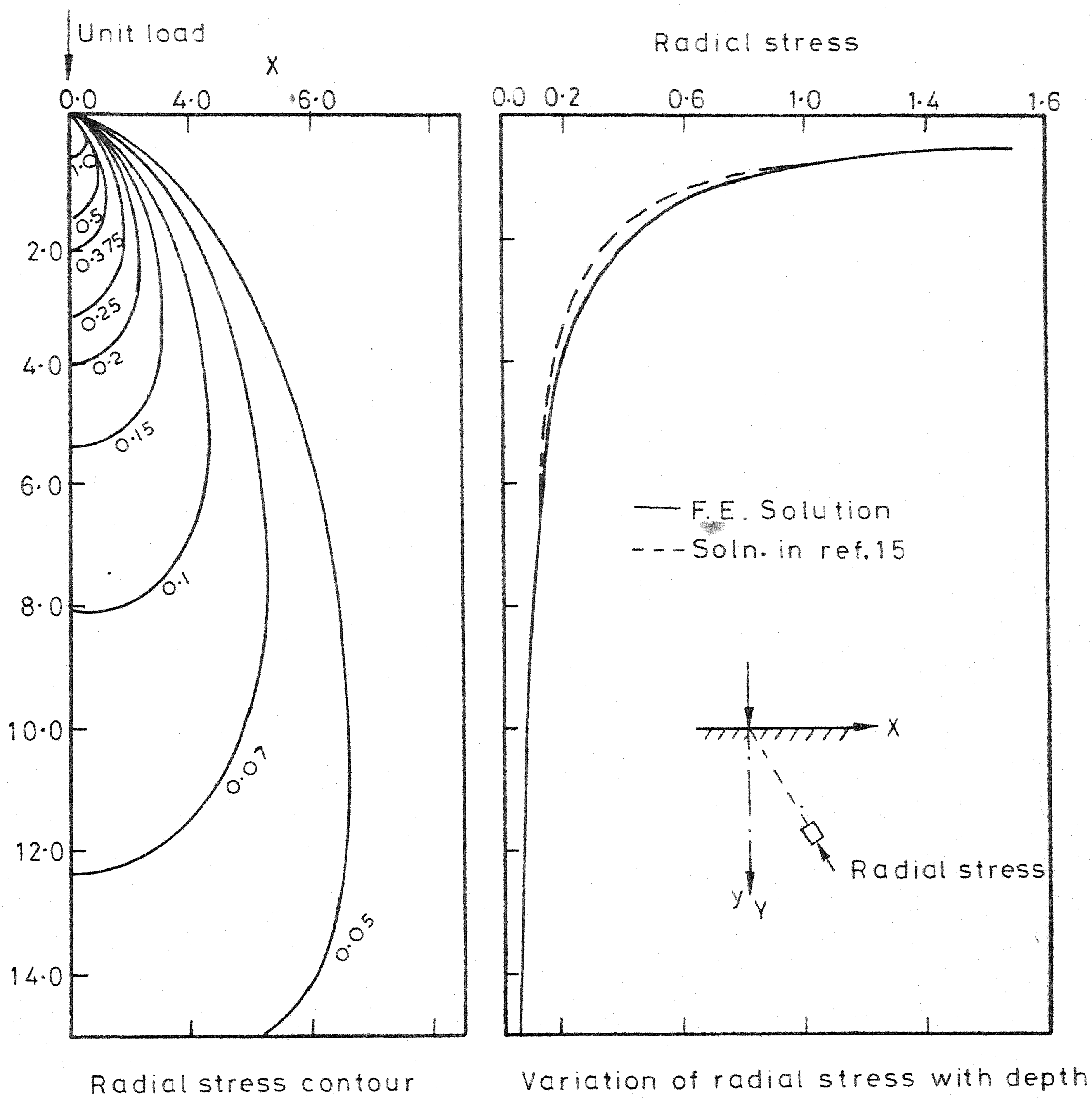


FIG. (3-4) STRESS DISTRIBUTION FOR HALF SPACE DUE TO A CONCENTRATED LOAD

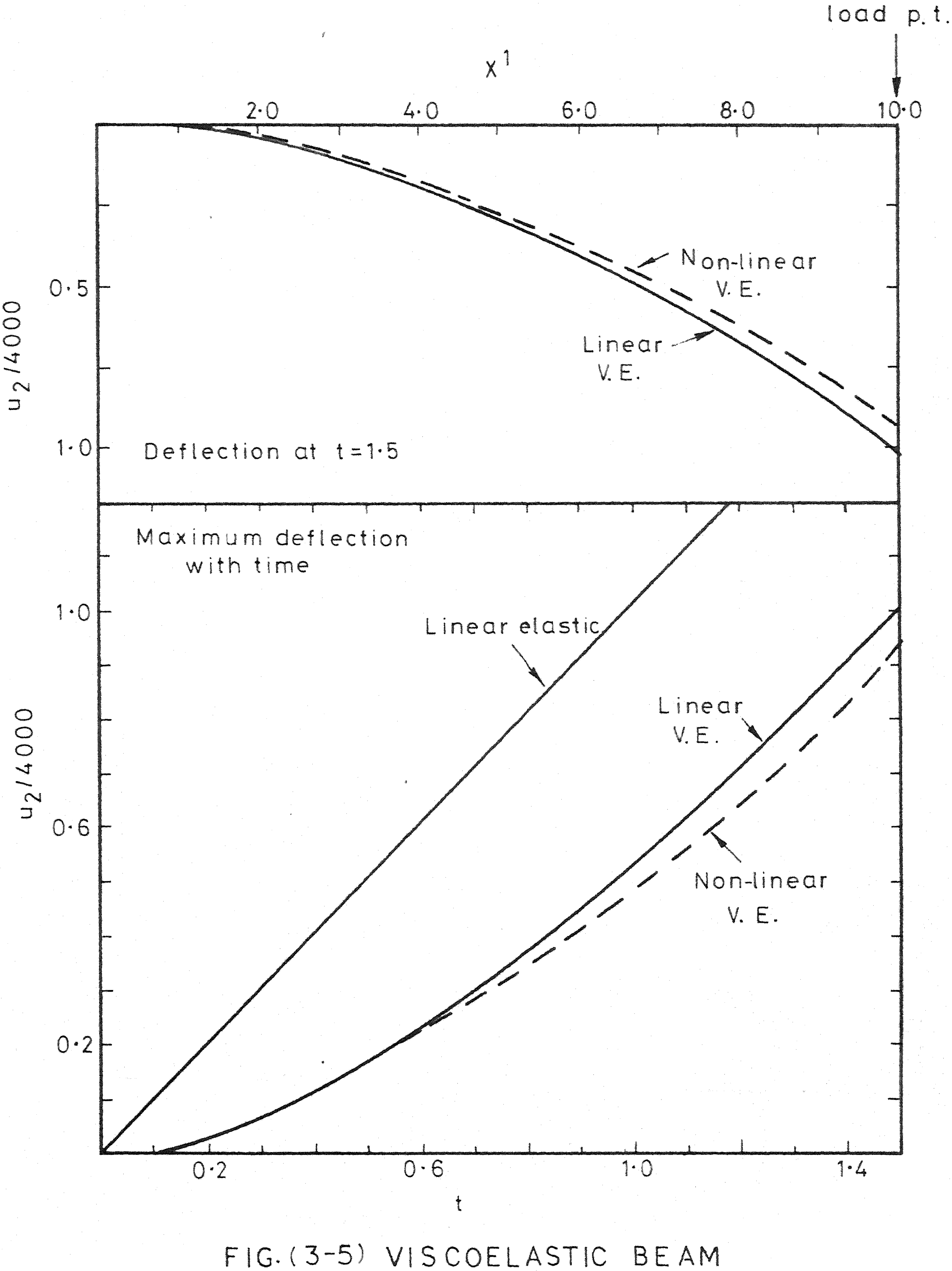


FIG. (3-5) VISCOELASTIC BEAM

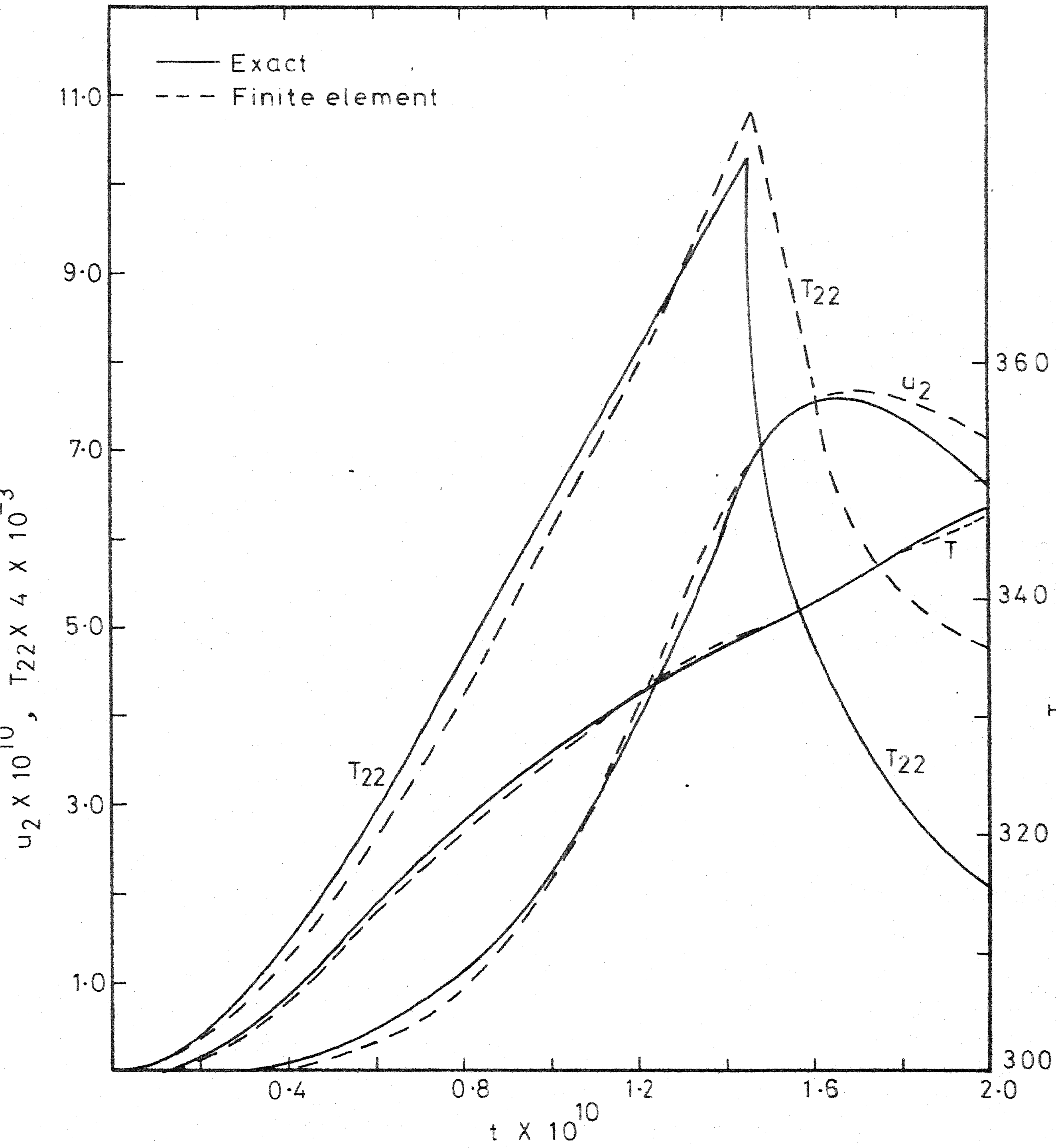


FIG. (3-6) THERMAL LOADING ON A ELASTIC HALF SPACE

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CHAPTER 4

AXISYMMETRIC SOLIDS

4.1 Introduction:

For deep sea exploration vessels and for aerospace structures, the analysis of complex axisymmetric bodies of arbitrary shape subjected to thermal and mechanical loads is of considerable interest. Because of the complex nature of the governing differential equations, the stress analyst must rely on numerical techniques. Although, the finite difference technique has been most popular, but it has its own limitations for nonhomogeneous materials and for arbitrary boundary conditions. Finite element method has already been applied for the analysis of these type of structures under suitable simplifications. Rashid and Clough[1] first applied this technique considering axisymmetric torous element having triangular cross-section with isotropic linear material. Wilson[2] has extended it to anisotropic material subjected to arbitrary mechanical and thermal loadings using fourier analysis, though for examples, he has mainly concentrated on axisymmetric loads. Recently Utku[3] has given explicit expressions of stiffness matrix for the triangular ring shaped element. It will be further extended here to accommodate nonlinear geometry and physical properties.

In the present chapter, the general governing equations (1.10-1) to (1.10-3) will be simplified in cylindrical coordinate system (Fig. 4-1) for materials having constitutive equations as functions of deformation velocity gradients and temperature. The temperature distribution has been assumed to be known as in the case of uncoupled thermoelastic problems, though the equation for thermal state with mechanical coupling can be derived without any difficulty. The basic motivation is to reduce the system to a two dimensional case by uncoupling the circumferential direction (θ-coordinate). This can be done by expanding the field variables (which are three displacements and known temperature distribution) in harmonic series in terms of θ. It will be seen that the most general anisotropic material which can be incorporated here shall have thirteen coefficients (see Section 108-109 of Reference 4). Moreover, for nonlinear cases, only axisymmetric deformations can be considered.

In Section two, the general equations are obtained for the nonlinear case. Section three is devoted for linear case, while in Section four nonlinear geometry with linear physical law is considered. Section five will provide a note for nonlinear material properties and in Section six, illustrative examples will be given.

4.2 General Equations and Notations:

Adopting single subscripted variable system for stresses and strains, and substituting T^{ij} from Eq. (1.10-2) in terms of T^{ij} in (1.10-1), the three governing equations can be **rewritten** in physical components for cylindrical polar coordinates. Thus,

$$\int_v \left(T_{1,1} + \frac{1}{r} T_{4,2} + T_{6,3} + \frac{T_1 - T_2}{r} + \rho_o (F_1 - f_1)\right)\dot{U}_1 dv$$

$$+ \int_s (P_1 - \bar{n}_1 T_1 - \bar{n}_2 T_4 - \bar{n}_3 T_6)\dot{U}_1' \, ds = 0$$

$$\int_v \left(T_{4,1} + \frac{1}{r} T_{2,2} + T_{5,3} + \frac{2T_4}{r} + \rho_o (F_2 - f_1)\right)\dot{U}_2 dv$$

$$+ \int_s (P_2 - \bar{n}_1 T_4 - \bar{n}_2 T_2 - \bar{n}_3 T_5) \dot{U}_2' \, ds = 0$$

$$\int_v \left(T_{6,1} + \frac{1}{r} T_{5,2} + T_{3,3} + \frac{T_6}{r} + \rho_o (F_3 - f_1)\right)\dot{U}_3 \, dv$$

$$+ \int_s (P_3 - \bar{n}_1 T_6 - \bar{n}_2 T_5 - \bar{n}_3 T_3) \dot{U}_3' \, ds = 0 \qquad (4.2\text{-}1)$$

where 1, 2 and 3 corresponds to directions r, θ and z respectively, P_i, F_i and f_i are boundary body and **inertial** forces respectively, $\bar{n}_i$ are direction cosines of the normal at a point on the boundary surface s, $\dot{U}_i'$ are corresponding velocities on the boundary s, the single subscripted stresses are,

$$\begin{array}{llllll} T_1 = \hat{T}_{11} & T_2 = \hat{T}_{22} & T_3 = \hat{T}_{33} \\ T_4 = \hat{T}_{12} & T_5 = \hat{T}_{23} & T_6 = \hat{T}_{13} \end{array} \qquad (4.2\text{-}2)$$

and $T_{1,1}$ represents the partial derivative of T_1 with respect to r. All quantities in Eq. (4.2-1) are physical components. Excepting the surface integral portions, Eqs. (4.2-1) are well known equations of equilibrium and are given in Section 59 of reference 4.

The constitutive Eqs.(1.10-6), can now be written as,

$$
\begin{aligned}
T_1 &= \hat{T}_1 + \hat{T}_1 U_{1,1} + \hat{T}_4 (U_{1,2} - U_2)/r + \hat{T}_6 U_{1,3} \\
T_2 &= \hat{T}_2 + \hat{T}_2 (U_{2,2} + U_1)/r + \hat{T}_4 U_{2,1} + \hat{T}_5 U_{2,3} \\
T_3 &= \hat{T}_3 + \hat{T}_3 U_{3,3} + \hat{T}_5 U_{3,2}/r + \hat{T}_6 U_{3,1} \\
T_4 &= \hat{T}_4 + \frac{1}{2} \Big| \hat{T}_1 U_{2,1} + \hat{T}_2 (U_{1,2} - U_2)/r \\
&\quad + \hat{T}_4 \{ U_{1,1} + (U_{2,2} + U_1)/r \} + \hat{T}_5 U_{1,3} + \hat{T}_6 U_{2,3} \Big| \\
T_5 &= \hat{T}_5 + \frac{1}{2} \Big| \hat{T}_2 U_{3,2}/r + \hat{T}_3 U_{2,3} + \hat{T}_4 U_{3,1} \\
&\quad + \hat{T}_5 \{ (U_{2,2} + U_1)/r + U_{3,3} \} + \hat{T}_6 U_{2,1} \Big| \\
T_6 &= \hat{T}_6 + \frac{1}{2} \Big| \hat{T}_1 U_{3,1} + \hat{T}_3 U_{1,3} + \hat{T}_4 U_{3,2}/r \\
&\quad + \hat{T}_5 (U_{1,2} - U_2)/r + \hat{T}_6 (U_{3,3} + U_{1,1}) \Big|
\end{aligned}
\qquad (4.2\text{-}3)
$$

where
$$\hat{T}_i = \hat{T}_i (\epsilon_1, \ldots, \epsilon_6, \gamma_1, \ldots \gamma_6, T) \qquad (4.2\text{-}4)$$

and $\epsilon_1, \ldots, \epsilon_6$ are classical strain, $\gamma_1, \ldots, \gamma_6$ are Rivlin-Ericksen tensors in modified form and T is the temperature. The classical nonlinear strains (1.10-9) are,

$$\epsilon_1 = U_{1,1} + ((U_{1,1})^2 + (U_{2,1})^2 + (U_{3,1})^2)/2$$

$$\epsilon_2 = (U_{2,2} + U_1)/r + ((U_{1,2}-U_2)^2 + (U_{2,2}+U_1)^2 + (U_{3,2})^2)/2r^2$$

$$\epsilon_3 = U_{3,3} + ((U_{1,3})^2 + (U_{2,3})^2 + (U_{3,3})^2)/2$$

$$2\epsilon_4 = (U_{1,2} - U_2)/r + U_{2,1} + (U_{1,1}(U_{1,2}-U_2) + U_{2,1}(U_{2,2} + U_1) + U_{3,1} U_{3,2})/r$$

$$2\epsilon_5 = U_{2,3} + U_{3,2}/r + (U_{1,3}(U_{1,2} - U_2) + U_{2,3}(U_{2,2}+U_1) + U_{3,3} U_{3,2})/r$$

$$2\epsilon_6 = U_{1,3} + U_{3,1} + U_{1,1} U_{1,3} + U_{2,1} U_{2,3} + U_{3,1} U_{3,3} \qquad (4.2\text{-}5)$$

and the Rivlin-Ericksen tensors (1.10-11) for cylindrical coordinate in physical form are,

$$\gamma_1 = \dot{U}_{1,1} \quad \gamma_2 = (\dot{U}_{2,2} + \dot{U}_1)/r \quad \gamma_3 = \dot{U}_{3,3}$$

$$2\gamma_4 = (U_{1,1}-\dot{U}_2)/r \quad 2\gamma_5 = \dot{U}_{2,3}+\dot{U}_{3,2}/r \quad 2\gamma_6 = \dot{U}_{1,3}+\dot{U}_{3,1} \qquad (4.2\text{-}6)$$

It may be noted that γ's are the time derivatives of the linear parts of ϵ's.

To derive the finite element equations for the axisymmetric solid, let the whole region be subdivided into R number of axisymmetric torous (ring shaped) element. Let each element have the shape of a polygon in radial section (Fig.4-2) and let N be the number of nodes ($N \geq 3$) per element. A typical

nodal point may be denoted by p and element by m. A particular node p may be common to s number of surrounding element. For an element m, the nodes are designated as 1,2,...,N and their coordinates (r_1, z_1), (r_2, z_2), ..., (r_N, z_N). Also let the displacements U_i (i = 1,2 and 3) be expressed in terms of their nodal values at N number of nodes through an interpolation law. Thus separating the variables, the displacements can be written in the form,

$$
\begin{aligned}
U_1(r, \theta, z, t) &= A_{mn}(t)\, D_m(r, z)\, H_n^1(\theta) \\
U_2(r, \theta, z, t) &= B_{mn}(t)\, D_m(r, z)\, H_n^2(\theta) \\
U_3(r, \theta, z, t) &= C_{mn}(t)\, D_m(r, z)\, H_n^1(\theta) \qquad (4.2\text{-}7) \\
\text{and} \quad T(r, \theta, z, t) &= E_{mn}(t)\, D_m(r, z)\, H_n^1(\theta)
\end{aligned}
$$

where $H_n^1(\theta) = \cos n\theta$ $H_n^2(\theta) = \sin n\theta$ and the repeated indices denote summation. The interpolation function D_m (m = 1, 2, ... N) have been assumed same for all the field variables. $A_{mn}(t)$, $B_{mn}(t)$, $C_{mn}(t)$ and $E_{mn}(t)$ are the nodal values of three displacements and temperature of the node m, at time t and for the nth Fourier expansion of the loading and the indeces are $m=1,2,..,N$; $n=0,1,..,\infty$.

The distribution for T is not very important, because it has been assumed that T is a known quantity. However, since this formulation can be directly coupled with a derivation for thermal state, and the combined set may be treated for coupled thermal equations, the above mentioned distribution for temperature will be adopted.

4.3 Linear Case:

Neglecting the nonlinear terms in (4.2-5), the linear strain expressions can be written as,

$$\epsilon_1^o = A_{mn} D_{m,1} H^1_n$$
$$\epsilon_2^o = (B_{mn} D_m n + A_{mn} D_m) H^1_n/r$$
$$\epsilon_3^o = C_{mn} D_{m,3} H^1_n$$
$$2\epsilon_4^o = -((A_{mn} D_m n + B_{mn} D_m)/r - B_{mn} D_{m,1}) H^2_n.$$
$$2\epsilon_5^o = (B_{mn} D_{m,3} - n C_{mn} D_m/r) H^2_n$$
$$2\epsilon_6^o = (A_{mn} D_{m,3} + C_{mn} D_{m,1}) H^1_n$$

For this simplified case, the constitutive relations (4.2-3) and (4.2-4) can now be expressed as,

$$T_i^o = \tilde{C}_{ij} \epsilon_j^o + \underset{\sim}{C}_{ij} \gamma_j + C_i T \quad i,j = 1,\ldots, 6 \quad (4.3\text{-}2)$$

where for linear material properties, the material coefficients $\tilde{C}_{ij}$, $\underset{\sim}{C}_{ij}$ and C_i are constants and are given by

$$\tilde{C}_{ij} = \frac{\partial \hat{T}_i}{\partial \epsilon_j} \quad ; \quad \underset{\sim}{C}_{ij} = \frac{\partial \hat{T}_i}{\partial \gamma_j} \quad ; \quad C_i = \frac{\partial \hat{T}_i}{\partial T} \quad .$$

and

$$\tilde{C}_{ij} = \tilde{C}_j \quad ; \quad \underset{\sim}{C}_{ij} = \underset{\sim}{C}_{ji}$$

From physical reasoning, it can be considered that,

$$C_i = 0, \quad i = 4, 5 \text{ and } 6 \quad (4.3\text{-}4)$$

From the strain expressions (4.3-1) and constitutive relations (4.3-2), it is evident that the stress components

will not be uncoupled unless,

$$\tilde{C}_{1j} = \tilde{C}_{2j} = \tilde{C}_{3j} = \tilde{C}_{6j} = 0 \qquad \text{for } j = 4,\ 5 \qquad (4.3\text{-}5)$$

and $\tilde{C}_{ij} = \tilde{C}_{ji}$

The constants $\underset{\sim}{C}_{ij}$ will have similar restrictions. Hence, altogether, there are thirteen constants from static consideration, thirteen constants from velocity consideration and three constants from temperature consideration.

Subjected to the above restrictions (4.3-5), the stress components in (4.3-2) can be subdivided as,

$$T_i^o = T_i^+ + T_i^- + T_i^x \qquad (4.3\text{-}6)$$

where,

$$T_i^+ = \tilde{C}_{ij}\, \epsilon_j^o \quad ; \quad T_j^- = \underset{\sim}{C}_{ij} \quad \gamma_j \qquad (4.3\text{-}7)$$

and $T_i^x = C_i T$

Now substituting strain expressions from (4.3-1) in $(4.3\text{-}7)_1$, T_i^+ can be written as,

$$T_i^+ = (\tilde{X}_{im}^1\, A_{mn} + \tilde{X}_{im}^2\, n\, B_{mn} + \tilde{X}_{im}^3\, C_{mn}) H_n^1$$

$$i = 1,2,3 \text{ and } 6 \qquad (4.3\text{-}8)$$

and $$T_i^+ = (-\tilde{X}_{im}^4\, n\, A_{mn} + \tilde{X}_{im}^5\, B_{mn} - \tilde{X}_{im}^6\, n\, C_{mn})\ H_n^2$$

$$i = 4,\ 5 \qquad (4.3\text{-}9)$$

where,

$$\tilde{X}_{im}^1 = \tilde{C}_{i1}\, D_{m,1} + \tilde{C}_{i2}\, D_m/r + \tilde{C}_{i6}\, D_{m,3}/2$$

$$\tilde{X}_{im}^2 = \tilde{C}_{i2}, D_m\, n/r$$

$$\tilde{X}^3_{im} = \tilde{C}_{i3}\, D_{m,3} + \tilde{C}_{i6}\, D_{m,1}/2$$

$$\tilde{X}^4_{im} = -\tilde{C}_{i4}\, D_m \;\; n/2r \qquad (4.3\text{-}10)$$

$$\tilde{X}^5_{im} = (\tilde{C}_{i4}\, (D_{m,1} - D_m/r) + \tilde{C}_{i5}\, D_{m,3})/2$$

$$\tilde{X}^6_{im} = -\, \tilde{C}_{i5}\, D_m/2r$$

Similarly, T^-_i can be written as,

$$T^-_i = (\underset{\sim}{X}^1_{im}\, \dot{A}_{mn} + \underset{\sim}{X}^2_{im}\, \dot{B}_{mn} + \underset{\sim}{X}^3_{im}\, \dot{C}_{mn})H^1_n$$

$$i = 1,2,3 \text{ and } 6 \qquad (4.3\text{-}11)$$

and $$T^-_i = (\underset{\sim}{X}^4_{im}\, \dot{A}_{mn} + \underset{\sim}{X}^5_{im}\, \dot{B}_{mn} + \underset{\sim}{X}^6_{im}\, \dot{C}_{mn})H^2_n$$

$$i = 4,\ 5 \qquad (4.3\text{-}12)$$

where $\underset{\sim}{X}^j_{im}$ can be obtained from $\tilde{X}^j_{im}$ in (4.3-10) by replacing $\tilde{C}_{ik}$ with $\underset{\sim}{C}_{ik}$. For example,

$$\underset{\sim}{X}^1_{im} = \underset{\sim}{C}_{i1}\, D_{m,1} + \underset{\sim}{C}_{i2}\, D_m/r + \underset{\sim}{C}_{i6}\, D_{m,3}/2 \qquad (4.3\text{-}13)$$

From $(4.3\text{-}7)_3$ and $(4.2\text{-}7)_4$, T^x_i can be written as,

$$T^x_i = X_{im}\, E_{mn}\, H^1_n \qquad (4.3\text{-}14)$$

where $$X_{im} = C_i\, D_m \qquad i = 1,\ 2 \text{ and } 3 \qquad (4.3\text{-}15)$$

Substituting the expressions for stress from (4.3-6), (4.3-8), (4.3-9), (4.3-11), (4.3-12) and (4.3-14) in (4.2-1), and using distributions (4.2-7), the first equilibrium equations can be written in the approximate form (noting $\bar{n}_2 = 0$, since it is a complete ring element),

$$\Sigma' \int_0^{2\pi} < \int_A | \{\tilde{X}^1_{1m,1} - \tilde{X}^4_{4m}\, n^2/r + \tilde{X}^1_{6m,3} + (\tilde{X}^1_{1m} - \tilde{X}^1_{2m})/r\}\, A_{mn}$$

$$+ n\{\, \tilde{X}^2_{1m,1} + \tilde{X}^5_{4m}/r + \tilde{X}^2_{6m,3} + (\tilde{X}^2_{1m} - \tilde{X}^2_{2m})/r\}\, B_{mn}$$

$$+ \{\tilde{X}^3_{1m,1} - \tilde{X}^4_{4m}\, n^2/r + \tilde{X}^3_{6m,3} + (\tilde{X}^3_{1m} - \tilde{X}^3_{2m})/r\} C_{mn}$$

$$+ \{\ \}\ \dot{A}_{mn} + \{\ \}\ \dot{B}_{mn} + \{\ \}\ \dot{C}_{mn} + \{X_{1m,1}$$

$$+ (X_{1m} - X_{2m})/r\}\ E_{mn} + \rho_o (F^n_1 - f^n_1)\ |\ D_p\ dA$$

$$+ \int_l \{P^n_1 - (\bar{n}_1 \tilde{X}^1_{1m} + \bar{n}_3 \tilde{X}^1_{6m}) A_{mn} - n(\bar{n}_1 \tilde{X}^2_{1m} + \bar{n}_3 \tilde{X}^2_{6m}) B_{mn}$$

$$- (n_1 \tilde{X}^3_{1m} + \bar{n}_3 \tilde{X}^3_{6m})\ C_{mn} - (\quad)\dot{A}_{mn} - (\ )\dot{B}_{mn} - (\quad)\dot{C}_{mn}$$

$$- (n_1 X_{1m}) E_{mn} \}\ D^1_p\ dl > H^1_n\ H^1_q\ \dot{A}_{pq}\ d\theta = 0$$

where D^1_p corresponds to boundary distribution, A represent area of cross-section and l, the element boundary for θ = constant. Since H^1_n and H^2_n are orthogonal functions such that,

$$\int_0^{2\pi} H^1_n\ H^1_p\ d\theta = \int_0^{2\pi} H^2_n\ H^2_p\ d\theta = \pi\ \delta_{np} \qquad (4.3\text{-}17)$$

the Eq. (4.3-15) will be completely uncoupled with respect to θ. The symbol Σ' denotes the summation over all the elements forming the entire axisymmetric solid. In similar manner the other two equations can be obtained and since equations are valid for arbitrary $\dot{A}_{pq}$, $\dot{B}_{pq}$ and $\dot{C}_{pq}$, (which are not equal to zero), the determinate number of equations will be obtained and can be solved for a particular n. Dropping the subscripts n, the three equations can be written in the

concise matrix form,

$$\Sigma' < | \underset{\sim}{K} |_{pm} \{ \dot{Z} \}_m + \{ \tilde{K} \}_{pm} \{Z\}_m + \{ K \}_{pm} \{E\}_m + \{L\}_p > = 0 \tag{4.3-18}$$

where,

$$\{ Z \} = \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix} \tag{4.3-19}$$

$$\{ E \}_m = E_{mn} \tag{4.3-20}$$

$$\{ K \}_{pm} = \begin{bmatrix} \underset{\sim}{K}_{11} & \underset{\sim}{K}_{12} & \underset{\sim}{K}_{13} \\ \underset{\sim}{K}_{21} & \underset{\sim}{K}_{22} & \underset{\sim}{K}_{23} \\ \underset{\sim}{K}_{31} & \underset{\sim}{K}_{32} & \underset{\sim}{K}_{33} \end{bmatrix}_{pm} \tag{4.3-21}$$

$$| K | = \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix}_{pm} \tag{4.3-22}$$

and $$\{L\} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}_p \tag{4.3-23}$$

The expressions for $\tilde{K}_{11}, \ldots, \tilde{K}_{66}$ are as follows:

$$\tilde{K}_{11} = \pi \,|\, \int_A (\tilde{X}^1_{1m,1} - \tilde{X}^4_{4m} n^2/r + \tilde{X}^1_{6m,3} + (\tilde{X}^1_{1m} - \tilde{X}^1_{2m})/r) D_p \, dA$$

$$- \int_{\underset{\sim}{l}} (n_1 \tilde{X}^1_{1m} + n_3 \tilde{X}^1_{6m}) D^1_p \, d\underset{\sim}{l} \,|$$

$$\tilde{K}_{12} = \bar{n}\pi \,\Big|\, \int_A (\tilde{X}^2_{1m} + \tilde{X}^5_{4m}/r + \tilde{X}^2_{6m,3} + (\tilde{X}^2_{1m} - \tilde{X}^2_{2m})/r) D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^2_{1m} + \bar{n}_3 \, \tilde{X}^2_{6m}) \, D^1_p \, dl \,\Big|$$

$$\tilde{K}_{13} = \pi \,\Big|\, \int_A (\tilde{X}^3_{1m} - \tilde{X}^6_{4m} \, \bar{n}^2/r + \tilde{X}^3_{6m,3} + (\tilde{X}^3_{1m} - \tilde{X}^3_{2m})/r) D_p \, dA$$

$$- \int_{l} (\bar{n}_1 \, X^3_{1m} + \bar{n}_3 \, X^3_{6m}) \, D^1_p \, dl \,\Big|$$

$$\tilde{K}_{21} = -n\pi \,\Big|\, \int_A (\tilde{X}^4_{4m,1} + \tilde{X}^1_{2m}/r + \tilde{X}^4_{5m,3} + 2\tilde{X}^4_{4m}/r) \, D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^4_{4m} + \bar{n}_3 \, \tilde{X}^4_{5m}) \, D^1_p \, d\tilde{l} \,\Big|$$

$$\tilde{K}_{22} = \pi \,\Big|\, \int_A (\tilde{X}^5_{4m,1} - \tilde{X}^2_{2m} \, n^2/r + \tilde{X}^5_{5m,3} + \frac{2}{r} \tilde{X}^5_{4m}) \, D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^5_{4m} + \bar{n}_3 \, \tilde{X}^5_{5m}) \, D^1_p \, d\tilde{l} \,\Big|$$

$$\tilde{K}_{23} = -n\pi \,\Big|\, \int_A (\tilde{X}^6_{4m,1} + \tilde{X}^3_{2m}/r + \tilde{X}^6_{5m,3} + 2\tilde{X}^6_{4m}/r) \, D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^6_{4m} + \bar{n}_3 \, \tilde{X}^6_{5m}) \, D^1_p \, d\tilde{l} \,\Big|$$

$$\tilde{K}_{31} = \pi \,\Big|\, \int_A (\tilde{X}^1_{6m,1} - \tilde{X}^4_{5m} \, n^2/r + \tilde{X}^1_{3m,3} + \tilde{X}^1_{6m}/r) \, D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^1_{6m} + \bar{n}_3 \, \tilde{X}^1_{3m}) \, D^1_p \, d\tilde{l} \,\Big|$$

$$\tilde{K}_{32} = n\pi \,\Big|\, \int_A (\tilde{X}^2_{6m,1} + \tilde{X}^5_{5m}/r + \tilde{X}^2_{3m,3} + \tilde{X}^2_{6m}/r) \, D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^2_{6m} + \bar{n}_3 \, \tilde{X}^2_{3m}) \, D^1_p \, d\tilde{l} \,\Big|$$

$$\tilde{K}_{33} = \pi \,\Big|\, \int_{\tilde{A}} (\tilde{X}^3_{6m,1} - \tilde{X}^6_{5m} \, n^2/r + \tilde{X}^3_{3m,3} + \tilde{X}^3_{6m}/r) \, D_p \, dA$$

$$- \int_{\tilde{l}} (\bar{n}_1 \, \tilde{X}^3_{6m} + \bar{n}_3 \, \tilde{X}^3_{3m}) \, D^1_p \, dl \,\Big|$$

Similarly, $\underset{\sim}{K}_{ij}$ can be obtained from $\tilde{K}_{ij}$ by replacing X by in (4.3-24). The expressions concerning the temperature effect are,

$$K_1 = \pi \left| \int_A (X_{1m,1} + X_{1m}/r)\, D_p dA - \int_{\underset{\sim}{l}} n_1\, X_{1m}\, D_p^1\, d\underset{\sim}{l} \right|$$

$$K_2 = -n\pi \left| \int_A X_{2m}/r\, D_p\, dA \right| \qquad (4.3\text{-}25)$$

$$K_3 = \pi \left| \int_A X_{3m,3}\, D_p\, dA - \int_{\underset{\sim}{l}} \bar{n}_3\, X_{3m}\, D_p^1\, d\underset{\sim}{l} \right|$$

Neglecting the inertia forces, the load vector is given by,

$$L_1 = \left| \int_A \rho_o\, F_1^n\, D_p\, dA + \int_{\underset{\sim}{l}} P_1^n\, D_p^1\, d\underset{\sim}{l} \right|$$

$$L_2 = \left| \int_A \rho_o\, F_2^n\, D_p\, dA + \int_{\underset{\sim}{l}} P_2^n\, D_p'\, d\underset{\sim}{l} \right| \qquad (4.3\text{-}26)$$

$$L_3 = \left| \int_A \rho_o\, F_3^n\, D_p\, dA + \int_{\underset{\sim}{l}} P_3^n\, D_p'\, d\underset{\sim}{l} \right|$$

where

$$(F_i^n,\, P_i^n) = \int_o^{2\pi} (F_i,\, P_i)\, H_n^1\, d\theta \qquad i = 1 \text{ and } 3$$

$$\text{and} \quad (F_2^n,\, P_2^n) = \int_o^{2\pi} (F_2,\, P_2)\, H_n^2\, d\theta \qquad (4.3\text{-}27)$$

Now, by choosing convenient numerical quadratures over the area, expressions (4.3-24) to (4.3-26) can be evaluated and then applying suitable integration technique, the matrix differential equation (4.3-18) can be solved.

It may be noted here that all the loading cases cannot be solved easily by the chosen displacement form (4.2-7). For example, the axisymmetric torsional (circumferential) load

needs special modifications. This type of problems can be solved easily if the distributions for U_i and T are chosen by interchanging H_n^1 and H_n^2, i.e.,

$$
\begin{aligned}
U_1\ (r,\ \theta,\ z,\ t) &= A_{mn}(t)\ D_m(r,\ z)\ H_n^2(\theta) \\
U_2\ (r,\ \theta,\ z,\ t) &= B_{mn}(t)\ D_m(r,\ z)\ H_n(\theta) \\
U_3\ (r,\ \theta,\ z,\ t) &= C_{mn}(t)\ D_m(r,\ z)\ H_n^2(\theta) \\
\text{and}\quad T\ (r,\ \theta,\ z,\ t) &= E_{mn}(t)\ D_m(r,\ z)\ H_n^2(\theta)
\end{aligned}
\tag{4.3-28}
$$

For this distribution, the modifications are as follows:

1. In Expressions (4.3-24) and (4.3-25) change n by -n.
2. Replace Expressions (4.3-27) by,

$$
\begin{aligned}
(F_i^n,\ P_i^n) &= \int_0^{2\pi} (F_i,\ P_i)\ H_n^2\ d\theta\ , \quad i = 1 \text{ and } 3 \\
\text{and}\quad (F_2^n,\ P_2^n) &= \int_0^{2\pi} (F_2,\ P_2)\ H_n^1\ d\theta
\end{aligned}
\tag{4.3-29}
$$

The remaining portion of the derivation will be unaltered.

4.4 Nonlinear Geometry:

It can be seen that if the distribution (4.2-7) is substituted in the nonlinear strain expressions (4.2-5), the θ-coordinate will not be uncoupled, unless it is assumed that nonlinearity in strains arises due to axisymmetric deformations, or in other words, the axisymmetric deformations are relatively much more prominent than those of nonaxisymmetric nature. This tantamount to the restrictions that,

$$
\begin{aligned}
(A_{pq},\ B_{pq},\ C_{pq}).(A_{mn}, B_{mn}, C_{mn}) \ll\ &(A_{po}, B_{po}, C_{po}) \\
&.(A_{mk},\ B_{mk},\ C_{mk})
\end{aligned}
\tag{4.4-1}
$$

where q and $n \neq 0$ and $k = 0, 1, 2, \ldots$
and the left hand quantities can be neglected when compared with the right hand quantities.

Under this suposition, the nonlinear strains can be written in the form,

$$\epsilon_i = \epsilon_i^o + \bar{\epsilon}_i \;, \quad i = 1, 2, \ldots, 6 \tag{4.4-2}$$

where ϵ_i^o are linear strains (4.3-1) and $\bar{\epsilon}_i$ are nonlinear parts which may be obtained by substituting the distributions (4.2-7) in (4.2-5) with the restrictions (4.4-1). Thus the incremental expressions for the strains are,

$$\bar{\epsilon}_1^* = ((A_{pn} \cdot A_{mo}^* + \delta' A_{mo} A_{pn}^*) + (C_{pn} \cdot C_{mo}^* + \delta\, C_{mo} C_{pn}^*))\, D_{p,1} D_{m,1} H_n^1$$

$$\bar{\epsilon}_2^* = \frac{1}{r^2}(A_{mo}^* (A_{pn} + n\, B_{pn}) + A_{mo}(A_{pn}^* + n\, B_{pn}^*)) D_m D_p H_n^1$$

$$\bar{\epsilon}_3^* = (C_{pn} C_{mo}^* + \delta' C_{mo} C_{pn}^* + A_{pn} A_{mo}^* + \delta' A_{mo} A_{pn}^*)\, D_{m,3} D_{p,3} H_n^1$$

$$\begin{aligned} 2\bar{\epsilon}_4^* = \frac{1}{r} | &(-(n\, A_{pn} + B_{pn}) D_{m,1} D_p + B_{pn} D_{p,1} D_m) A_{mo}^* \\ &- A_{mo} (n\, A_{pn}^* D_{m,1} D_p + B_{pn}^*(D_{m,1}D_p - D_m D_{p,1})) \\ &- n\, (C_{mo} C_{pn}^* + C_{pn} C_{mo}^*) D_{m,1} D_p | H_n^2 \end{aligned} \tag{4.4-3}$$

$$2\bar{\epsilon}_5^* = \frac{1}{r} \left| (-(nA_{pn} + B_{pn}) D_{m,3} D_p + B_{pn} D_{p,3} D_m) A^*_{mo} \right.$$

$$- A_{mo} (n A^*_{pn} D_{m,3} D_p + B^*_{pn} (D_{m,3} D_p - D_{p,3} D_m))$$

$$\left. - n (C_{mo} C^*_{pn} + C_{pn} C^*_{mo}) D_{m,3} D_p \right| H_n^2$$

$$2\bar{\epsilon}_6^* = \left| (\delta' A^*_{mo} A_{pn} + A^*_{pn} A_{mo} + \delta' C^*_{mo} C_{pn} + C^*_{pn} C_{mo}) \right.$$

$$\left. (D_{p,3} D_{m,1} + D_{m,3} D_{p,1}) \right|$$

where $\delta' = 0$ when $n = 0$ otherwise $\delta' = 1$.

Now neglecting any third power of nodal displacements, the incremental stresses can be written in the form

$$T_i^* = T_i^{o*} + \bar{T}_i^* \qquad i = 1, \ldots, 6 \tag{4.4-4}$$

where T_i^{o*} is the linear part and are obtained from (4.3-6) and (4.3-7). The expressions for $\bar{T}_i^*$ can be obtained from (4.2-3) by using the linear stresses (4.3-8), (4.3-9), (4.3-11), (4.3-12) and (4.3-14) and the nonlinear part of strains (4.4-3). Thus following the restrictions (4.4-1) and neglecting the terms which are more than second order in nodal displacements, the nonlinear part of the incremental stresses can be written in the form,

$$\bar{T}_i^* = \left| G^{1n}_{im} A^*_{mo} + G^{3n}_{im} C^*_{mo} + \tilde{G}^1_{ip} A^*_{pn} + \tilde{G}^2_{ip} B^*_{pn} \right.$$

$$+ G^3_{ip} C^*_{pn} + \tilde{\underset{\sim}{X}}^{1n}_{im} E^*_{mo} + \tilde{\underset{\sim}{X}}^2_{ip} E^*_{pn} + \underset{\sim}{G}^{1n}_{im} \dot{A}^*_{mo}$$

$$\left. + \underset{\sim}{G}^{3n}_{im} \dot{C}^*_{mo} + \tilde{\underset{\sim}{G}}^1_{ip} \dot{A}^*_{pn} + \tilde{\underset{\sim}{G}}^2_{ip} \dot{B}^*_{pn} + \tilde{\underset{\sim}{G}}^3 \dot{C}^*_{pn} \right| (H^1_n, H^2_n)$$

$$(4.4\text{-}5)$$

where the index i varies from 1 to 6. The multiplier in the braces in (4.4-5) is H_n^1 when i = 1, 2, 3 and 6 and H_n^2 when i = 4 and 5. The symbols used in (4.4-5) are as follows;

$$
\begin{aligned}
G_{1m}^{1n} &= \tilde{Y}_{1m}^{1n} + \delta' \left| \tilde{X}_{1m}^{1} D_{p,1} + \tilde{X}_{6m}^{1} D_{p,3} \right| A_{pn} \\
&\qquad + \delta' \left| \hat{T}_1^n D_{m,1} + \hat{T}_6^n D_{m,3} \right| \\
G_{1m}^{3n} &= \tilde{Y}_{1m}^{3n} + \delta' \left| \tilde{X}_{1m}^{3} D_{p,1} + \tilde{X}_{6m}^{3} D_{p,3} \right| A_{pn} \\
\tilde{G}_{1p}^{1} &= \tilde{I}_{1p}^{1} + \left| \tilde{X}_{1p}^{1} D_{m,1} + \tilde{X}_{6p}^{1} D_{m,3} \right| A_{mo} \\
&\qquad + \left| \hat{T}_1^o D_{p,1} + \hat{T}_6^o D_{p,3} \right| \\
\tilde{G}_{1p}^{2} &= \tilde{I}_{1p}^{2} + n \left| \tilde{X}_{1p}^{2} D_{m,1} + \tilde{X}_{6p}^{2} D_{m,3} \right| A_{mo} \\
\tilde{G}_{1p}^{3} &= \tilde{I}_{1p}^{3} + \left| \tilde{X}_{1p}^{3} D_{m,1} + \tilde{X}_{6p}^{3} D_{m,3} \right| A_{mo} \\
\underset{\sim}{\tilde{X}}{}_{1m}^{1n} &= \delta' X_{1m} D_{p,1} A_{pn} \\
\underset{\sim}{\tilde{X}}{}_{1p}^{2} &= X_{1p} D_{m,1} A_{mo} \qquad\qquad (4.4\text{-}6) \\
\underset{\sim}{G}{}_{1p}^{1n} &= \delta' \left| \underset{\sim}{X}{}_{1m}^{1} D_{p,1} + X_{6m} D_{p,3} \right| A_{pn} \\
\underset{\sim}{G}{}_{1m}^{3n} &= \delta' \left| \underset{\sim}{X}{}_{1p}^{1} D_{m,1} + \underset{\sim}{X}{}_{6m}^{3} D_{p,3} \right| A_{pn} \\
\underset{\sim}{\tilde{G}}{}_{1p}^{1} &= \left| \tilde{X}_{1p}^{1} D_{m,1} + X_{6p} D_{m,3} \right| A_{mo} \\
\underset{\sim}{\tilde{G}}{}_{1p}^{2} &= n \left| X_{1p}^{2} D_{m,1} + X_{6p}^{2} D_{m,3} \right| A_{mo} \\
\underset{\sim}{\tilde{G}}{}_{1p}^{3} &= \left| \underset{\sim}{X}{}_{1p}^{3} D_{m,1} + \tilde{X}_{6p}^{3} D_{m,3} \right| A_{mo} \\
G_{2m}^{1n} &= \tilde{Y}_{2m}^{1n} + \delta' \left| \tilde{X}_{2m}^{1} \left(n B_{pn} D_p + A_{pn} D_p \right) \right| \\
&\qquad + \delta' \left| \hat{T}_2^n / r \; D_m \right|
\end{aligned}
$$

$$G^{3n}_{2m} = \tilde{Y}^{3n}_{2m} + \delta' | \tilde{X}^{3}_{2m} \cdot (n\, B_{pn}\, D_p + A_{pn}\, D_p) |$$

$$\tilde{G}^{1}_{2p} = \tilde{I}^{1}_{2p} + | \tilde{X}^{1}_{2p} \frac{D_m}{r} A_{mo} | + | \frac{\hat{T}^{o}_{2}}{r} D_p |$$

$$\tilde{G}^{2}_{2p} = \tilde{I}^{2}_{2p} + | X^{2}_{2p} \frac{D_m}{r} A_{mo} | + | n \frac{\hat{T}^{o}_{2}}{r} D_p | \qquad (4.4\text{-}7)$$

$$\tilde{G}^{3}_{2p} = \tilde{I}^{3}_{2p} + | \tilde{X}^{3}_{2p} \frac{D_m}{r} A_{mo} |$$

$$\underset{\sim}{\tilde{X}}^{1n}_{2m} = \delta'' (n\, B_{pn}\, D_p + A_{pn}\, D_p)\, X_{2m} \; ; \; \underset{\sim}{\tilde{X}}^{2}_{2p} = X_{2p} \frac{D_m}{r} A_{mo}$$

$$G^{1n}_{3m} = \tilde{Y}^{1n}_{3m} + \delta' | (\tilde{X}^{1}_{3m}\, D_{p,3} + \tilde{X}^{1}_{6m}\, D_{p,1})\, C_{pm} |$$
$$+ | \hat{T}^{N}_{3}\, D_{m,3} + \hat{T}^{N}_{6}\, D_{m,1} |$$

$$G^{3n}_{3m} = \tilde{Y}^{3n}_{3m} + \delta' | (\tilde{X}^{3}_{3m}\, D_{p,3} + X^{3}_{6m}\, D_{p,1})\, C_{pn} |$$

$$\tilde{G}^{1}_{3p} = I^{1}_{3p} + | (\tilde{X}^{1}_{3p}\, D_{m,3} + \tilde{X}^{1}_{6p}\, D_{m,1})\, C_{mo} |$$

$$\tilde{G}^{2}_{3p} = I^{2}_{3p} + | (\tilde{X}^{2}_{3p}\, D_{m,3} + \tilde{X}^{2}_{6p}\, D_{m,1})\, C_{mo} |$$

$$\tilde{G}^{3}_{3p} = I^{3}_{3p} + | (\tilde{X}^{3}_{3p}\, D_{m,3} + \tilde{X}^{3}_{6p}\, D_{m,1})\, C_{mo} |$$

$$\underset{\sim}{\tilde{X}}^{1n}_{3m} = \delta'\, X_{3m}\, D_{p,3}\, C_{pn}$$

$$\underset{\sim}{\tilde{X}}^{2}_{3p} = X_{3p}\, D_{m,3}\, C_{mo}$$

$$G^{1n}_{4m} = \tilde{Y}^{4n}_{4m} + \frac{1}{2} | (X^{1}_{1m}\, D_{p,1} + \tilde{X}^{1}_{6m}\, D_{p,3})\, B_{pn} |$$
$$+ \frac{1}{2} | \hat{T}^{n}_{4} (D_{m,1} + \frac{D_m}{r}) + \hat{T}^{n}_{5}\, D_{m,3} |$$

$$G^{3n}_{4m} = \tilde{Y}^{6n}_{4m} + \frac{1}{2} | (\tilde{X}^{3}_{1m}\, D_{p,1} + \tilde{X}^{3}_{6m}\, D_{p,3})\, B_{pn} |$$

$$\tilde{G}^1_{4p} = \tilde{I}^4_{4p} + \frac{1}{2} \left| \left(\left(D_{m,1} + \frac{D_m}{r} \right) \tilde{X}^4_{4p} + D_{m,3} \, \tilde{X}^4_{5p} \right) A_{mo} \right|$$

$$\tilde{G}^2_{4p} = I^5_{4p} + \frac{1}{2} \left| \left(\left(D_{m,1} + \frac{D_m}{r} \right) \tilde{X}^5_{4p} + D_{m,3} \, \tilde{X}^5_{5p} \right) A_{mo} \right|$$

$$+ \frac{1}{2} \left| \hat{T}^o_1 \, D_{p,1} + \hat{T}^o_6 \, D_{p,3} \right| \qquad (4.4\text{–}9)$$

$$\tilde{G}^3_{4p} = I^6_{4p} + \frac{1}{2} \left| \left(\left(D_{m,1} + \frac{D_m}{r} \right) \tilde{X}^6_{4p} + D_{m,3} \, \tilde{X}^6_{5p} \right) A_{mo} \right|$$

$$\tilde{X}^{1n}_{4m} = \frac{1}{2} \cdot X_{1m} \, D_{p,1} \, B_{pn}$$

$$G^{1n}_{5m} = \tilde{Y}^{4n}_{5m} + \frac{1}{2} \left| -n \, \frac{D_p}{r} \, \tilde{X}^1_{2m} \, C_{pn} + \left(\tilde{X}^1_{3m} \, D_{p,3} \right. \right.$$

$$\left. \left. + \tilde{X}^1_{6m} \, D_{p,1} \right) B_{pn} \right| + \frac{1}{2} \left| \hat{T}^n_4 \, D_{m,1} + \frac{1}{r} \, T^n_5 \, D_m \right|$$

$$G^{3n}_{5m} = \tilde{Y}^{6n}_5 + \frac{1}{2} \left| -n \, \tilde{X}^3_{2m} \, \frac{D_p}{r} \, C_{pn} + \left(\tilde{X}^3_{3m} \, D_{p,3} + \tilde{X}^3_{6m} \, D_{p,1} \right) B_{pn} \right|$$

$$+ \frac{1}{2} \left| -n \, \hat{T}^n_2 \, D_m + \hat{T}^n_5 \, D_{m,3} \right|$$

$$(4.4\text{–}10)$$

$$\tilde{G}^1_{5p} = \tilde{I}^4_{5p} + \frac{1}{2} \left| \tilde{X}^4_{4p} \, D_{m,1} \, C_{mo} + \left(\frac{D_m}{r} \, A_{mo} + D_{m,3} \, C_{mo} \right) \tilde{X}^4_{5p} \right|$$

$$\tilde{G}^2_{5p} = \tilde{I}^5_{5p} + \frac{1}{2} \left| \tilde{X}^5_{4p} \, D_{m,1} \, C_{mo} + \left(\frac{D_m}{r} \, A_{mo} + D_{m,3} \, C_{mo} \right) \tilde{X}^5_{5p} \right|$$

$$+ \frac{1}{2} \left| \hat{T}^o_6 \; D_{p,1} + \hat{T}^o_3 \, D_{p,3} \right|$$

$$\tilde{G}^3_{5p} = \tilde{I}^6_{5p} + \frac{1}{2} \left| \tilde{X}^6_{4p} \, D_{m,1} \, C_{mo} + \left(\frac{D_m}{r} \, A_{mo} + D_{m,3} \, C_{mo} \right) \tilde{X}^6_{5p} \right|$$

$$\tilde{X}^{1n}_{5m} = \frac{1}{2} \left| -n \, D_p \, X_{2m} \, C_{pn} \right| \frac{1}{r} + \frac{1}{2} \left| D_{m,3} \, X_{3p} \, B_{mn} \right|$$

$$G^{1n}_{6m} = \tilde{Y}^{1n}_{6m} + \frac{1}{2}\, \delta' \left| (\tilde{X}^{1}_{1m}\, D_{p,1} + \tilde{X}^{1}_{6m}\, D_{p,3})\, C_{pn} + (\tilde{X}^{1}_{3m}\, D_{p,3} + \tilde{X}^{1}_{6m}\, D_{p,1})\, A_{pn} \right| + \frac{1}{2}\, \delta' \left| \hat{T}^{n}_{1}\, D_{m,3} + \hat{T}^{n}_{6}\, D_{m,1} \right|$$

$$G^{3n}_{6m} = \tilde{Y}^{3n}_{6m} + \frac{1}{2}\, \delta' \left| (\tilde{X}^{3}_{1m}\, D_{p,1} + \tilde{X}^{3}_{6m}\, D_{p,3})\, C_{pn} + (\tilde{X}^{3}_{3m}\, D_{p,3} + \tilde{X}^{3}_{6m}\, D_{p,1})\, A_{pn} \right| + \frac{1}{2}\, \delta' \left| \hat{T}^{n}_{1}\, D_{m,1} + \hat{T}^{n}_{6}\, D_{m,3} \right|$$

$$\tilde{G}^{1}_{6p} = \tilde{I}^{1}_{6p} + \frac{1}{2} \left| (\tilde{X}^{1}_{1p}\, D_{m,1} + \tilde{X}^{1}_{6p}\, D_{m,3})\, C_{mo} + (\tilde{X}^{1}_{3p}\, D_{m,3} + \tilde{X}^{1}_{6p}\, D_{m,1})\, A_{mo} \right| + \frac{1}{2} \left| \hat{T}^{o}_{3}\, D_{m,3} + \hat{T}^{n}_{6}\, D_{m,1} \right|$$

$$(4.4\text{-}11)$$

$$\tilde{G}^{2}_{6p} = \tilde{I}^{2}_{6p} + \frac{1}{2} \left| (\tilde{X}^{2}_{1}\, D_{m,1} + \tilde{X}^{2}_{6p}\, D_{m,3})\, C_{mo} + (\tilde{X}^{2}_{3p}\, D_{m,3} + \tilde{X}^{2}_{6p}\, D_{m,1})\, A_{mo} \right|$$

$$\tilde{G}^{3}_{6p} = \tilde{I}^{3}_{6p} + \frac{1}{2} \left| (\tilde{X}^{3}_{1}\, D_{m,1} + \tilde{X}^{3}_{6p}\, D_{m,3})\, C_{mo} + (\tilde{X}^{3}_{3p}\, D_{m,3} + \tilde{X}^{3}_{6p}\, D_{m,1})\, A_{mo} \right| + \frac{1}{2} \left| \hat{T}^{o}_{1}\, D_{m,1} + \hat{T}^{o}_{6}\, D_{m,3} \right|$$

$$\underset{\sim}{X}^{1n}_{6m} = \frac{1}{2}\, \delta' \left| X_{1m}\, D_{p,1}\, C_{pn} + X_{3m}\, D_{p,3}\, A_{pn} \right|$$

$$\underset{\sim}{X}^{2}_{6p} = \frac{1}{2}\, \delta' \left| X_{1p}\, D_{m,1}\, C_{mo} + X_{3p}\, D_{m,3}\, A_{mo} \right|$$

In the expressions (4.4-6) to (4.4-11), δ' is zero whenever $n = 0$ otherwise it is one. The symbols $\tilde{X}^{k}_{im}$ and $\underset{\sim}{X}^{k}_{im}$ have been already defined in (4.3-10) and (4.3-13). The expressions for $\underset{\sim}{G}^{kn}_{im}$ and $\tilde{\underset{\sim}{G}}^{k}_{ip}$ may be obtained by considering only the curly bracket portion of the corresponding G^{km}_{im} and $\tilde{G}^{k}_{ip}$ respectively and by replacing $\tilde{X}^{k}_{im}$ by $\underset{\sim}{X}^{k}_{im}$. A typical

example has been given in(4.4-6). The expressions for $\tilde{Y}^{kn}_{im}$ and $\tilde{I}^{k}_{ip}$ are as given below,

$$\tilde{Y}^{1n}_{im} = \delta\tilde{C}_{i1}\, D_{m,1}\, D_{p,1}\, A_{pn} + \tilde{C}_{i2}\, D_m D_p (A_{pn} + n\, B_{pn})/r$$
$$+ \delta'\, \tilde{C}_{i3} \cdot D_{m,3}\, D_{p,3}\, A_{pn} + \delta'\, \tilde{C}_{i6}(D_{p,3}\, D_{m,1}$$
$$+ D_{m,3}\, D_{p,1})\, A_{pn}/2 \qquad (4.4\text{-}12)$$

$$\tilde{Y}^{3n}_{im} = \delta'(\tilde{C}_{i1}\, D_{m,1}\, D_{p,1} + \tilde{C}_{i3}\, D_{m,3}\, D_{p,3} + \tilde{C}_{i6}\, (D_{p,3}\, D_{m,1}$$
$$+ D_{m,3}\, D_{p,1})/2)\, C_{pn} \qquad (4.4\text{-}13)$$

$$\tilde{I}^{1}_{ip} = (\tilde{C}_{i1}\, D_{m,1}\, D_{p,1} + \tilde{C}_{i2}\, D_m\, D_p/r^2 + \tilde{C}_{i3}\, D_{m,3}\, D_{p,3}$$
$$+ \tilde{C}_{i6}\, (D_{p,3}\, D_{m,1} + D_{m,3}\, D_{p,1})/2)\, A_{mo} \qquad (4.4\text{-}14)$$

$$\tilde{I}^{2}_{ip} = n \,.\, \tilde{C}_{i2}\, D_m\, D_p\, A_{mo}/r^2 \qquad (4.4\text{-}15)$$

$$\tilde{I}^{3}_{ip} = (\tilde{C}_{i1}\, D_{m,1}\, D_{p,1} + \tilde{C}_{i3}\, D_{m,3}\, D_{p,3} + \tilde{C}_{i6}\, (D_{p,3}$$
$$+ D_{m,1} + D_{m,3}\, D_{p,1})/2)\, C_{mo} \qquad (4.4\text{-}16)$$

The index i in (4.4-12) to (4.4-16) takes the value 1,2,3 and 6. Correspondingly for i = 4 and 5. The following expressions are valid.

$$\tilde{Y}^{4n}_{im} = \frac{1}{2r}\Big|(-(n.A_{pn} + B_{pn})\, D_{m,1}\, D_p + B_{pn}\, D_{p,1}\, D_m)\, \tilde{C}_{i4}$$
$$+ (-(n.A_{pn} + B_{pn})\, D_{m,3}\, D_p + B_{pn}\, D_{m,3}\, D_m)\, \tilde{C}_{i5}\Big|$$
$$(4.4\text{-}17)$$

$$\tilde{Y}^{6n}_{im} = \frac{-n}{2r}\, \big|\, \tilde{C}_{i4}\, D_{m,1}\, D_p + \tilde{C}_{i5}\, D_{m,3}\, D_p \,\big|\, C_{pn} \qquad (4.4\text{-}18)$$

$$\tilde{I}^4_{ip} = \frac{n}{2r} \,|\, \tilde{C}_{i4}\ D_{m,1}\ D_p + \tilde{C}_{i5}\ D_{m,3}\ D_p \,|\, A_{mo} \qquad (4.4\text{-}19)$$

$$\tilde{I}^5_{ip} = \frac{1}{2r} \,|\, \tilde{C}_{i4}\ (D_{m,1}\ D_p + D_{p,1}\ D_m) + \tilde{C}_{i5}\ (D_{m,3} D_p + D_{p,3}\ D_m) \,|\, A_{mo} \qquad (4.4\text{-}20)$$

$$\tilde{I}^6_{ip} = \frac{-n}{2r} \,|\tilde{C}_{i4}\ D_{m,1}\ D_p + \hat{C}_{i5}\ D_{m,3}\ D_p \,|\, C_{mo} \qquad (4.4\text{-}21)$$

The stresses (4.4-5) together with T^{o*}_i from the linear analysis can now be substituted in the Eqs.(4.2-1) and subsequently solved using quadratures for the area and line integrations using numerical integration technique for solving the resulting differential equations in terms of nodal field variables. The expressions, as such, are extremely massive. But a great deal of simplification will be achieved if the distributions (4.2-7) are assumed to be linear.

It is worth mentioning here that this particular case is valid for the nonlinear stability analysis of an axisymmetric solid subjected to an axisymmetric load. Snap through and bifurcation phenomena can also be studied using these equations.

4.5 Nonlinear Material Property:

It can be clearly seen that non-axisymmetric deformation cannot be uncoupled even for small deformations, when physical nonlinearity is accounted for. It is only possible for the case of axisymmetric deformations. Material nonlinearity will not pose any additional difficulty in this case, except

the coefficients $\tilde{C}_{ij}$, $\underset{\sim}{C}_{ij}$ and C_i will no longer be constants, but will be functions of ϵ_{ij}, γ_{ij} and T subjected to the restrictions (4.3-5) at each point.

4.6 Applications:

4.6.1: For the solution of problems which has been illustrated in this section, consider a triangular ring element with linear distribution of displacements U_1, U_2 and U_3 without any temperature effect, i.e., $T(r, \theta, z, t)$ may be assumed to be constants. For this particular case N will be equal to 3 and for an element, the nodes will be 1, 2 and 3 and their coordinates (r_1, z_1), (r_2, z_2) and (r_3, z_3) as shown in Fig. (4-2).

Now, since the displacements are assumed linear over a triangular element, the interpolation functions are given by,

$$\langle D_1 \; D_2 \; D_3 \rangle = \langle 1 \quad r \quad z \rangle \, | g | \tag{4.6-1}$$

$$|g| = \frac{1}{2A} \begin{vmatrix} 2A/3 & 2A/3 & 2A/3 \\ z_2-z_3 & z_3-z_1 & z_1-z_2 \\ r_3-r_2 & r_1-r_3 & r_2-r_1 \end{vmatrix} \tag{4.6-2}$$

and $$2A = (r_2-r_1)(z_3-z_1) - (r_3-r_1)(z_2-z_1) \tag{4.6-3}$$

Using the interpolation functions (4.6-1) in Section 3 and 4 different cases can be solved through the use of computers.

Example 1:

Axisymmetric deformation of a linear elastic cylinder has been analyzed and compared with the analytical results from the text of A.E.H. Love[4]. The end cross-sections of the cylinder has been prevented from longitudinal movements but are free in other directions. The radial displacement U_1 and stresses T_1 and T_2 agree excellently with the analytical results. The longitudinal normal stress T_3 and its mean value 1.667 has been found to be in good agreement. In all the cases the error is less than 1 percent. For plotting the stresses, the average value has been calculated between an element and its adjacent inverted one within each square unit. The results have been shown in Fig. (4-3).

Example 2:

The same cylinder but now in the form of a thin disc has been analyzed considering the pressure variation as $\cos(n\theta)$ for different values of n (e.g., n = 2, 4, 6 and 10). As a typical example the displacements and stress distributions for n = 4 have been shown in Fig. (4-4) and (4-5). For analytical results, a small program has been made based on the derivation given in Timoshenko's text[5]. For displacements the error never exceeds 0.5 percent, but for stresses the discrepencies may be as high as 3 percent particularly if the displacement U_1 has a very flat curve. This has been found for the stress T_1. This has happened, possibily because of the low values of the stress which, in turn, has caused ill conditioning.

Example 3:

The problem of a rotating anisotropic disc has been solved and compared with analytical results given by Sam Tang[6] in Fig. (4-6). The dimensions of the disc and the arrangement of elements are same as the previous example. Magnesium has been chosen for the anisotropic material and its elastic constants are listed in Fig. (4-6). Here, it is only necessary to put the body force equal to the inertia force $\rho_o \omega^2 r$, where ρ_o is the mass per unit volume of the disc and ω is the angular velocity. The results agree excellently.

Example 4:

An anisotropic sphere made of Magnesium and subjected to internal and external shear load is shown in Fig. (4-7). This problem has been analyzed by Chen[7] and is compared with the finite element solution. Since the material symmetry is in spherical polar coordinate, the elastic constants have to be transformed to cylindrical coordinate. This is done at the center of gravity of each element by a slight modification of the program. The shear loads at the inner and outer surfaces and the circumferential displacements U_3 at the angular distance 67.5, 78.5 and 90.0 degrees from the apex have been plotted in the Fig. (4-7).

All the above four problems are linear and require only one step for solution. In multistep analysis, e.g., for non-linear or viscoelastic case, the available computer time

becomes the most stringent limitation. For this reason only two problem has been solved for nonlinear and viscoelastic case.

Example 5:

To investigate the nonlinear aspect of the model, an annular plate has been analyzed. The dimensions of the plate, its material constants and arrangement of the elements are shown in Figure (4-8). Initially the problem was solved by taking double the element sizes, shown in the figure. It has been observed that if the step for incremental load is reduced from 2 (which has been shown in Figure by the symbol X) to 1, the solution deviates away from the firm line representing the nonlinear solution for thin plate as given by Gordom B.J.Mah[8]. This is possibly because of compensation of errors due to discretisation and large step integration which has resulted for the step increment 2. The variation of displacement with load for different incremental steps, clearly shows the importance of proper step sizes. However, with the step of 1, quite satisfactory results have been obtained upto the loading point 5, although it can not be stated very definitely because the analytical results are based on thin plate theory.

Example 6:

A viscoelastic cylinder stiffened by an external elastic encasing with ablating inner surface has been analyzed by the finite element method and compared with analytical solution

presented by E.C.Ting[9]. The dimensions of the cylinder and the element subdivitions have been shown in Fig. (4-9A). The Young's modulus and the poisson's ratio of the outer elastic encasing are respectively $10.\times 10^5$ and 0.25. Now if $a(t)$ is the inner radius at time t such that $a(0)=5$ and since the outer radius of inner shell b is 10, the ablating rate from the reference 9 has been taken as,

$$a^2(t) = a^2(0)/\left|1 - (1-a^2(0)/b^2)t\right|$$

$$= 25/(1-3t/4) \quad , \quad t < 1$$

and the pressure variation with respect to time has been assumed as,

$$p(t) = (1 - e^{-5t})$$

The constitutive relation for the viscoelastic material has been prescribed as,

$$(T_1, T_2) = \int_0^t J(t - t') \left|\dot{\epsilon}_1 + \dot{\epsilon}_2\right| dt' + \int_0^t K(t - t') \left|\dot{\epsilon}_1, \dot{\epsilon}_2\right| dt'$$

and

$$T_3 = \int_0^t j(t - t') \left|\dot{\epsilon}_1 + \dot{\epsilon}_2\right| dt'$$

where $K(t)$ and $J(t)$ for this problem are given by,

$$K(t) = 82.0 + 9282.0\, e^{-1.126t}$$

and $$J(t) = 12472.7 - 3094.0 e^{-1.126t}$$

Due to ablation of the inner surface, at every new integration step, the arrangement of the elements have to be modified. This is done by the reducing the inner most column of elements and interpolating the previous values of elements and interpolating the previous values of strains at the center of gravity of new reduced elements. The integrations necessary for obtaining the material coefficients have been achieved by trapezoidal rule. It may be observed from the curve C of Figure (4-9) that stresses and displacement at the interface are agreeable upto t = 0.6. After this, the solution starts deviating away from the actual solution as shown by firm lines in the figures. Also the stress distributions are quite satisfactory except near the interface.

4.6.2: Comments:

From the above analyses, it has been found that one step solution is not at all a problem. But for multistep analysis and particularly for nonlinear geometry, the element configuration and the step sizes might be critical controlling factors. Since for the initial steps, a fairly conservative step length produces quite satisfactory result, it is essential that for subsequent steps a thorough check has to be made. Obviously for a problem with known result it is possible, but for unknown problems checking the solution by reducing the step sizes seems only possible.

Unfortunately it is a time consuming affair. However, Davidenko[10] has shown that the convergency can be much improved by employing more efficient integrating technique such as Runge-Kutta. Although, it is a complicated situation, but in the long run it may be more economic.

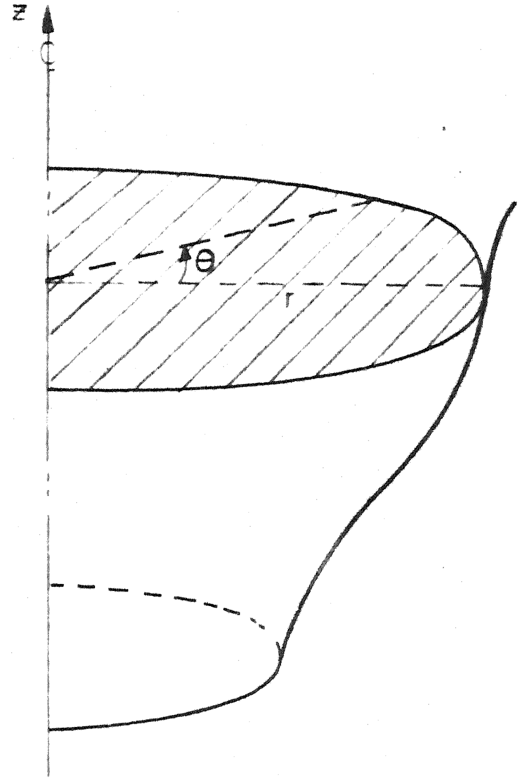


FIG (4-1) CYLINDRICAL COORDINATE

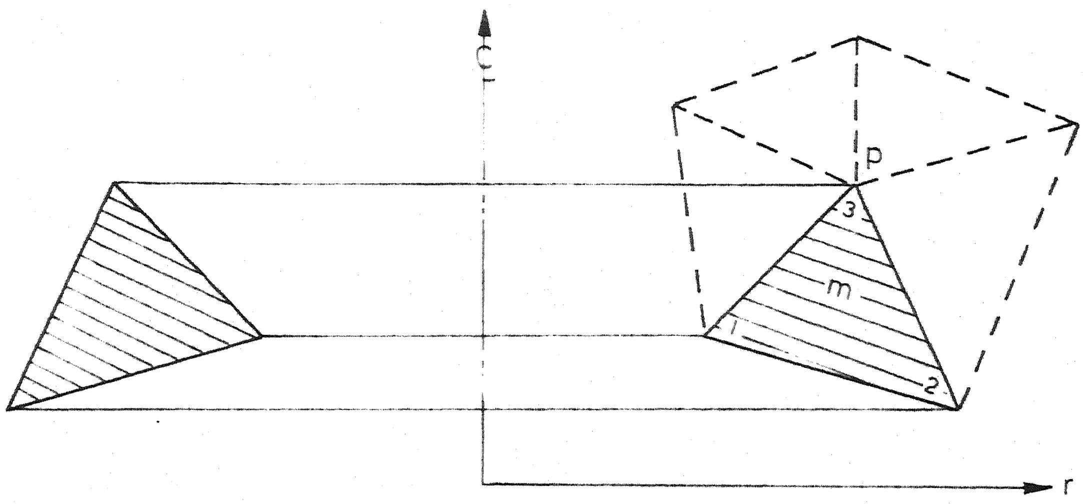


FIG. (4-2) ARRANGEMENT OF ELEMENT (n_o=3)

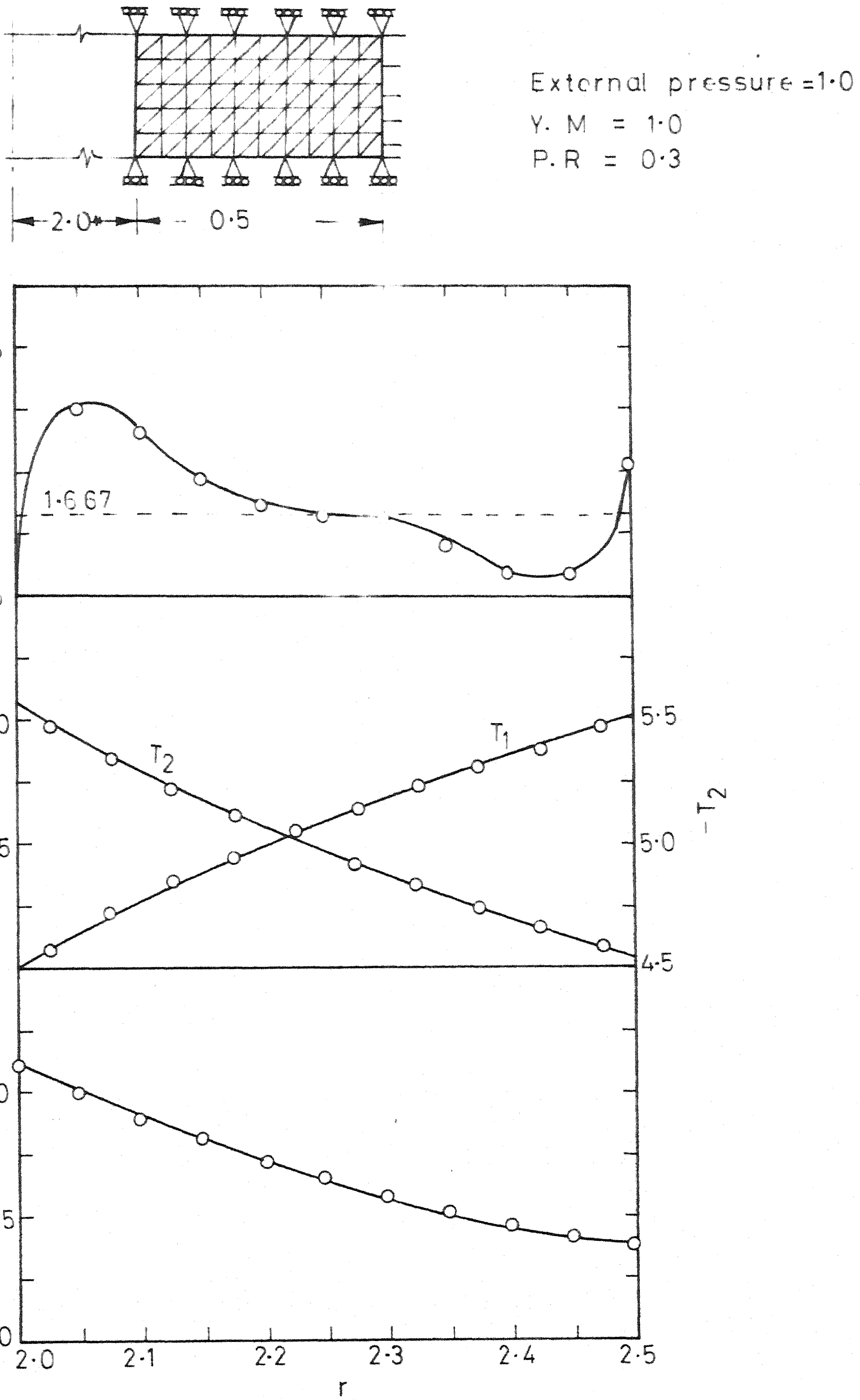


FIG.(4-3) STRESSES AND DISPLACEMENT OF AN INFINITE CYLINDER (n=0)

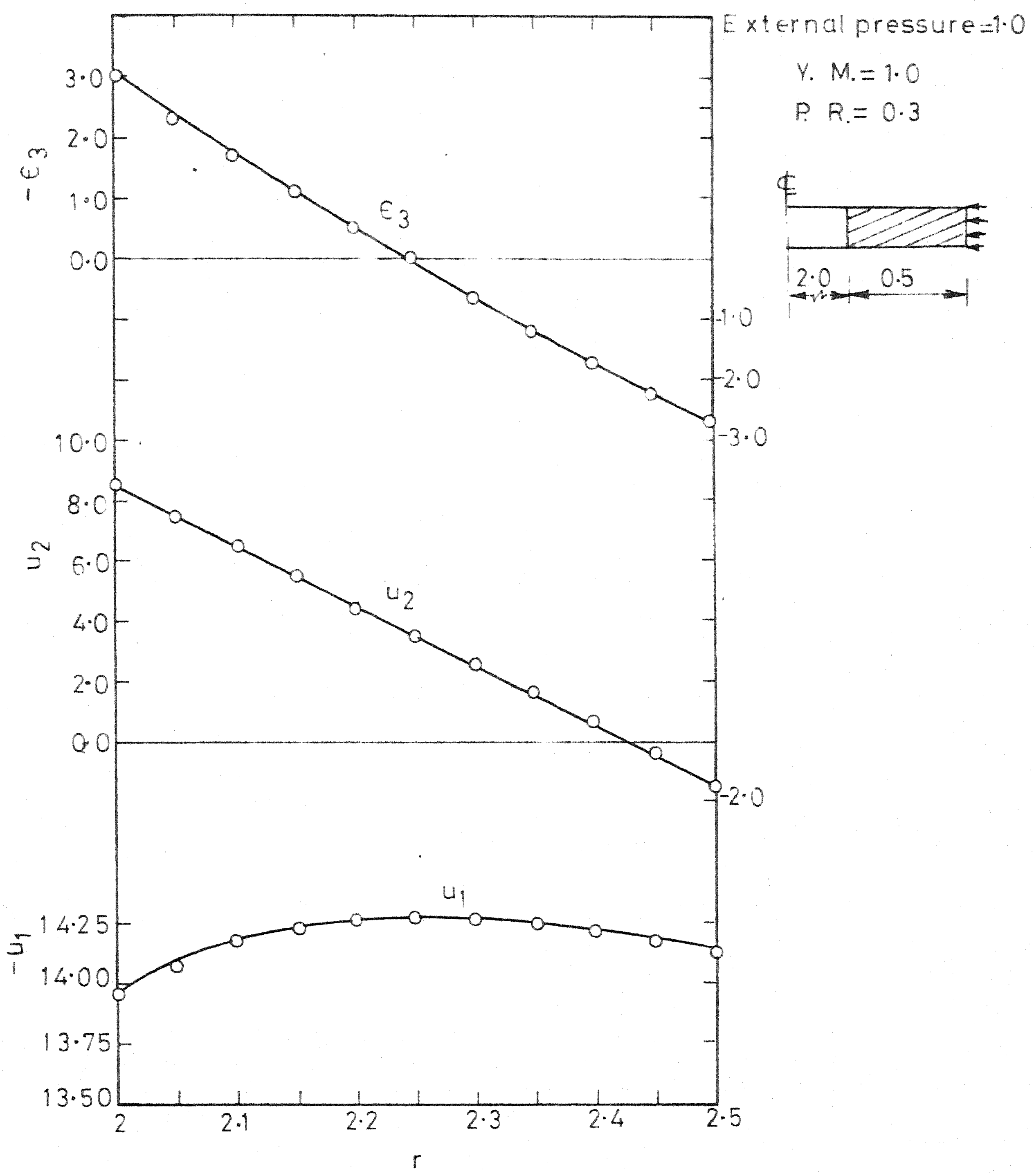


FIG. (4-4) DISPLACEMENTS AND STRAIN IN A CYLINDER (n= 4)

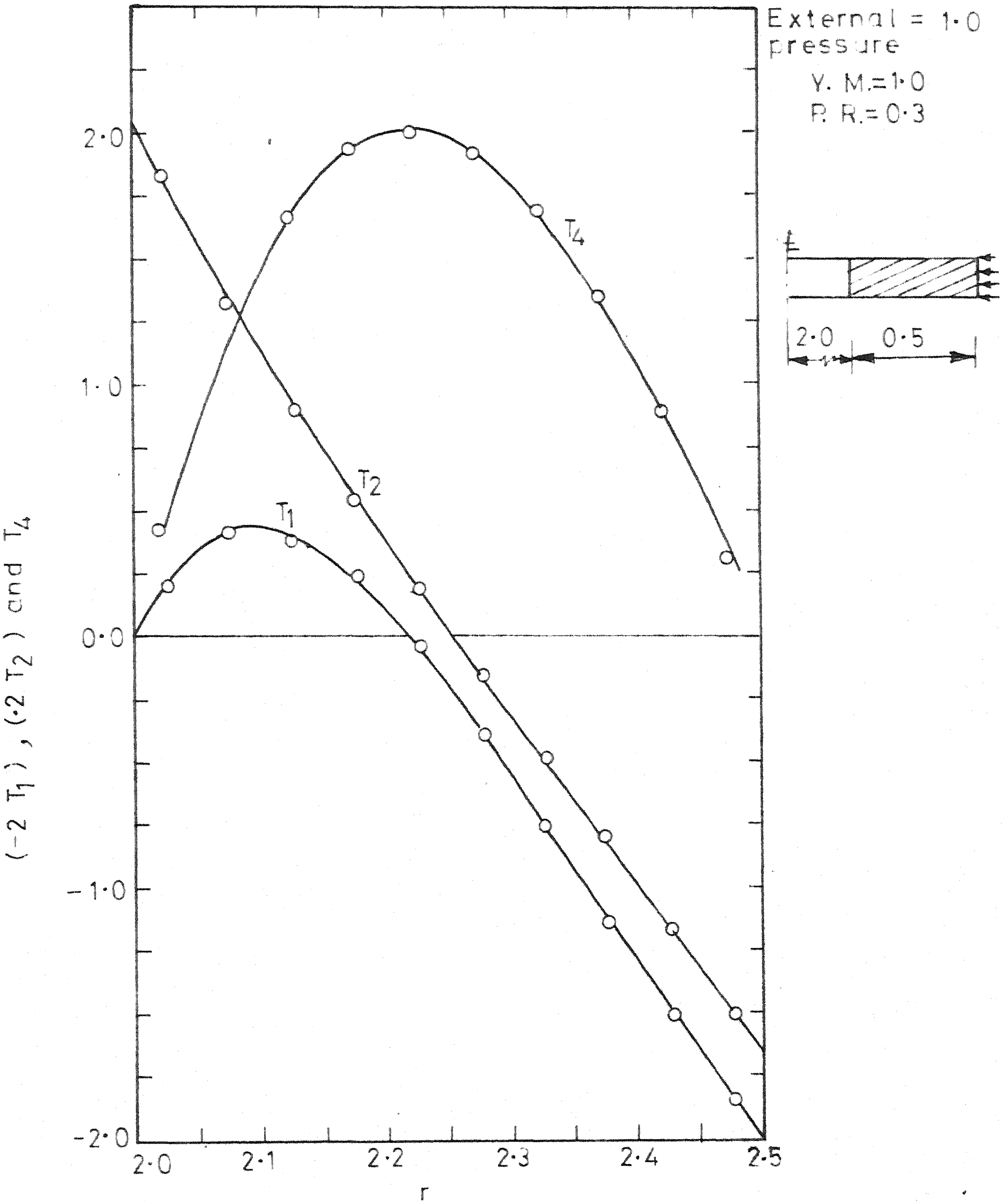
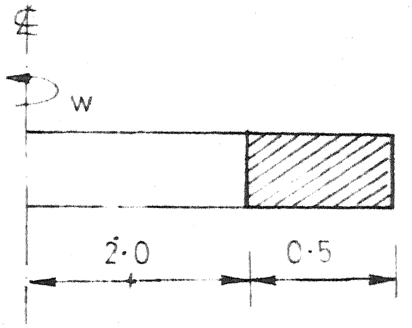


FIG. (4-5) STRESSES IN A CYLINDER (n=4)



$\rho w = 1 \cdot 0$

Material constant for magnesiu
(in cyl polar coordinate)

$C_{11} = 6 \cdot 17$

$C_{22} = C_{33} = 5 \cdot 97$

$C_{23} = 2 \cdot 62$

$C_{12} = C_{13} = 2 \cdot 17$

$C_{55} = 1 \cdot 67$

$C_{44} = C_{66} = 1 \cdot 64$

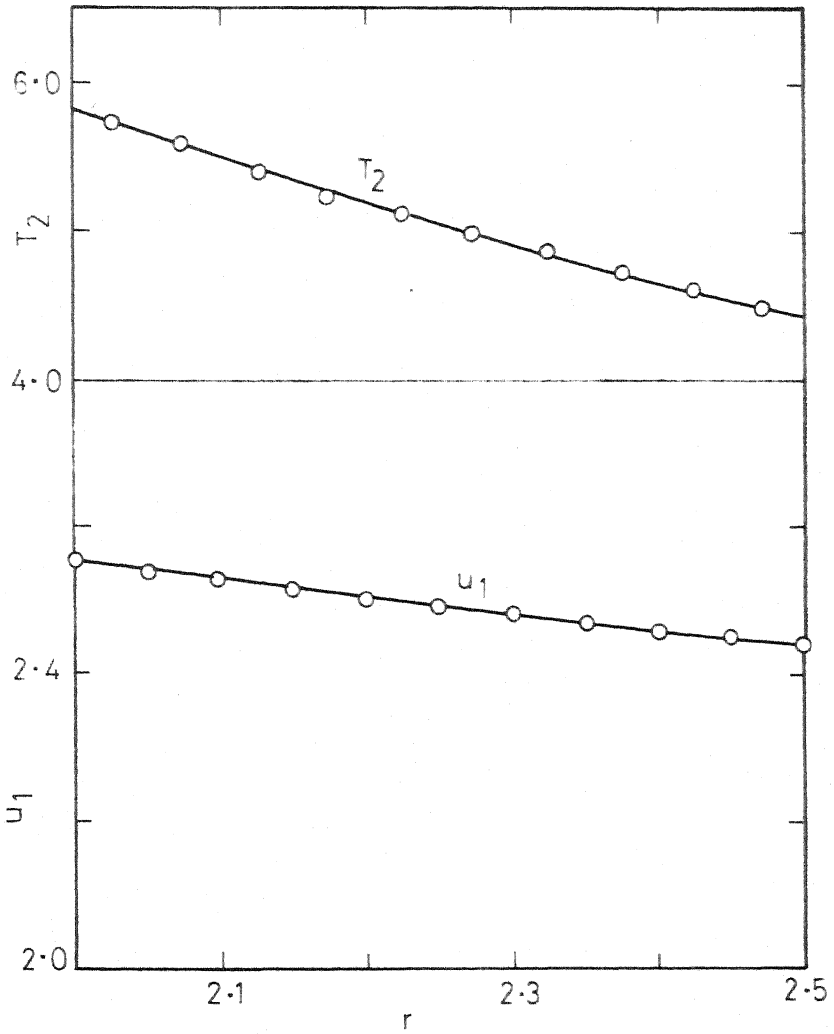


FIG. (4-6) DISPLACEMENT AND STRESS OF AN ANISOTROPIC ROTATING DISC.

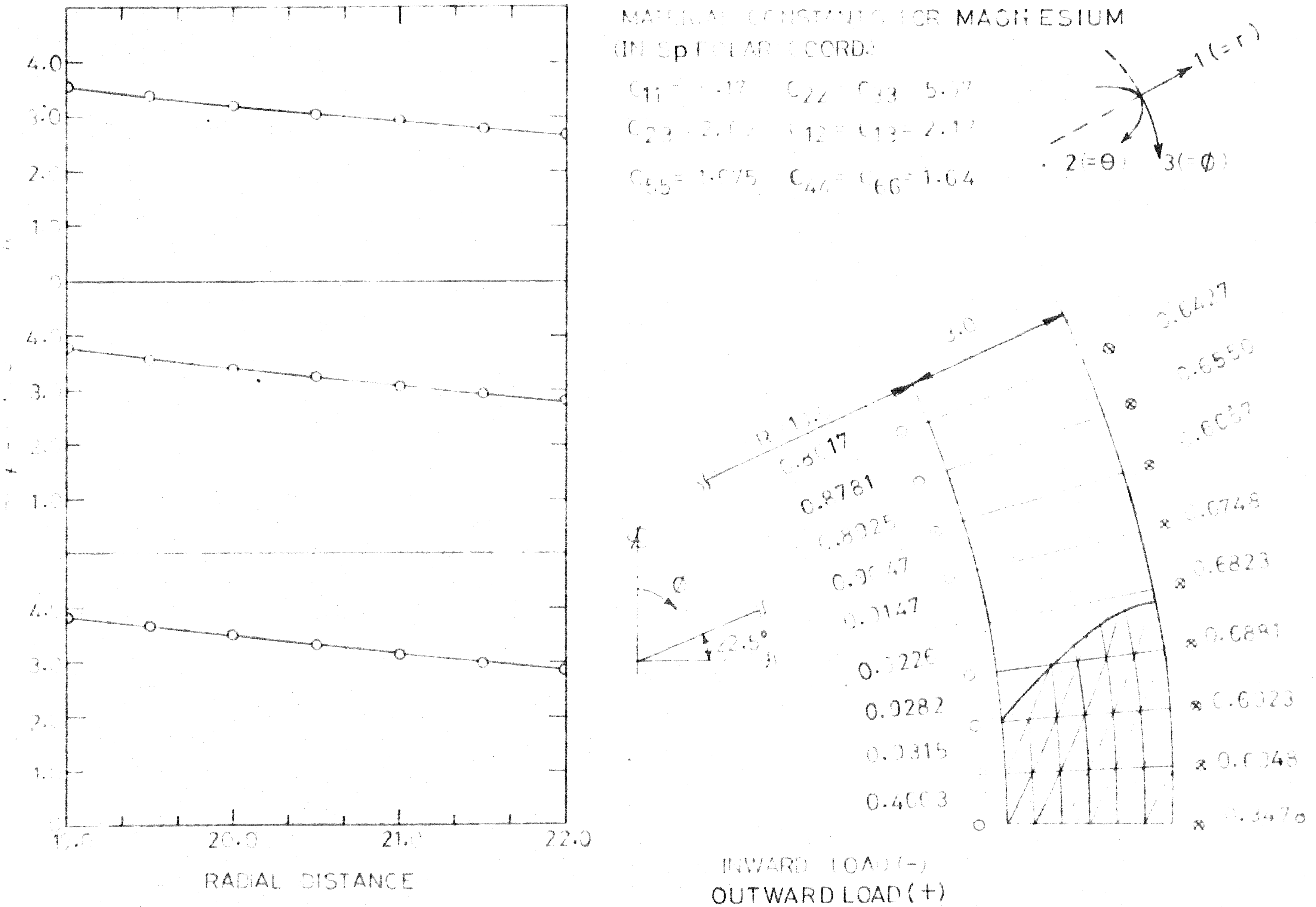
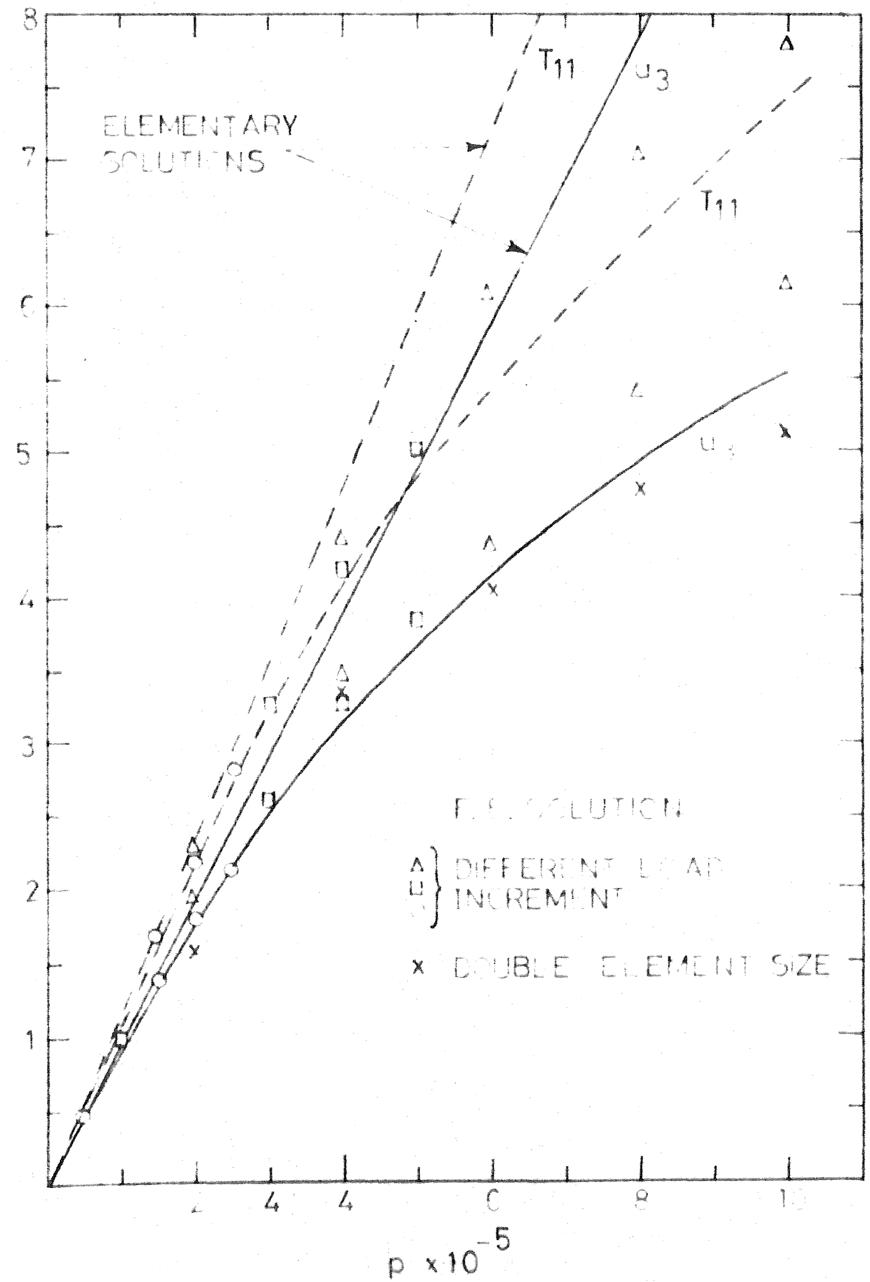
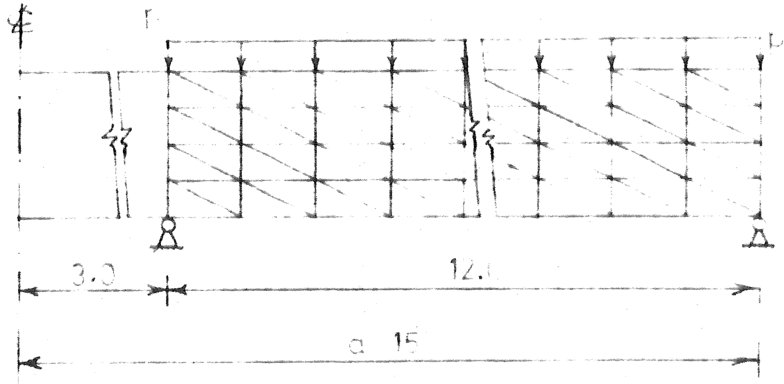


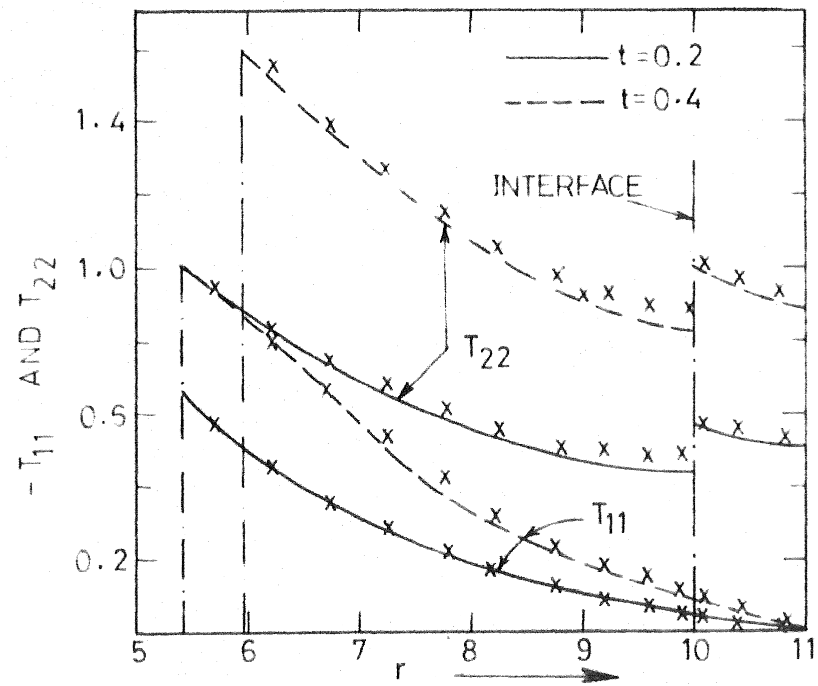
FIG.(4 - 7) ANISOTROPIC SPHERE WITH SHEAR LOAD



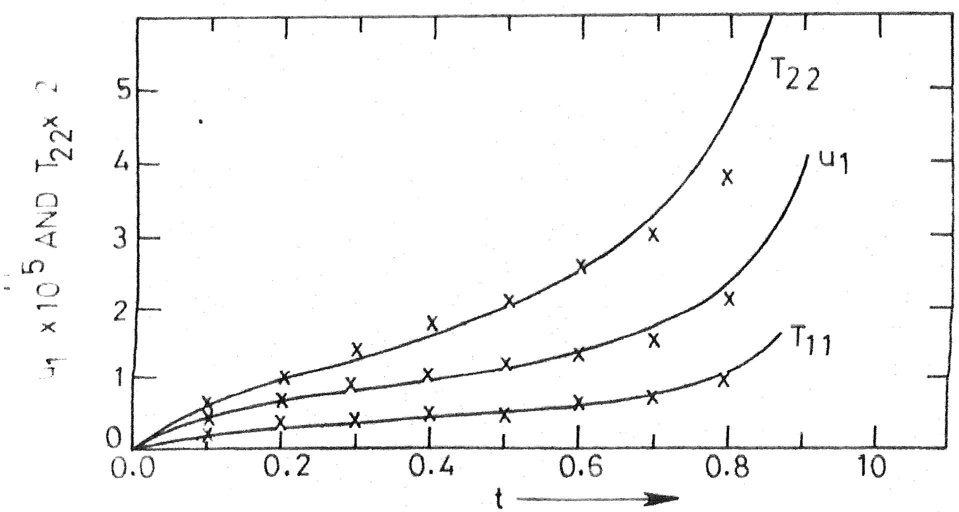


MATERIAL PROPERTY
Y.M. 1.
P.R. .3

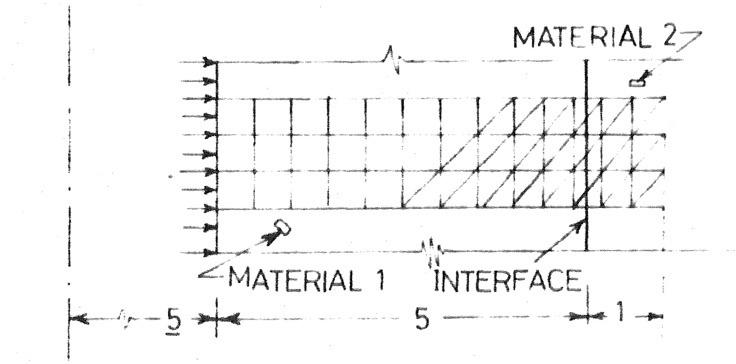
FIG.(4-8) NON-LINEAR ANALYSIS OF ANNULAR PLATE



B. STRESS DISTRIBUTION IN CYLINDER



C. HISTORY AT INTERFACE



A. INITIAL ARRANGEMENT OF ELEMENTS

FOR V.E. MATERIAL 1 AT t = 0

Y.M. = 1.25×10^4, P. R. 0.25

FOR ELASTIC ENCASING

Y.M. = 10.0×10^4 P. R. = 0.25

x FINITE ELEMENT SOLUTION

——— - - - - EXACT

FIG. (4-9) ANALYSIS OF VISCOELASTIC ENCASED CYLINDER

4.7 References:

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CHAPTER 5

TORSION ANALYSIS

5.1 Introduction:

It has been recognised that stress analysis of a non-circular prismatic member of nonlinear material properties subjected to pure torsion presents formidable mathematical difficulties even in the simplest cases. Closed form solutions exist for shafts having only a certain class of stress strain law and mainly for circular and a few other cross sections. Energy approach coupled with mathematical or physical discretization techniques may be applied for a more wider class of problems. In this chapter, the finite element technique has been applied for the analysis of torsion of prismatic members having arbitrary cross-section and varied nature of nonlinear material properties.

In Section 2, a brief literature survey has been presented. In Section 3, the complete form of three dimensional stress functions, introduced by Gurtin, has been shown to be identical to that of Prandtl for the torsion analysis. In the next Section, finite element approach has been employed for a prismatic member whose resultant stress and strain relation can be prescribed by a continuous (e.g. arc sinh , Ramberg-Osgood etc.) functions. The cross-section of the bar has been divided into a finite number of triangular elements,

Within these elements, the stress function $\emptyset$, which satisfies the equilibrium condition, is expressed in terms of $\emptyset$ alone or intterms of $\emptyset$ and its derivatives at the nodes, which in general, is termed as field variables. The complementary energy is then expressed in terms of the unknown field variables and using the principle of minimum complementary energy, a set of simultaneous nonlinear algebraic equations are obtained. For a given value of the angle of twist θ, these nonlinear equations are solved by step by step method in conjunction with an iterative scheme. In subsequent two Sections, examples are given for a triangular model with a linear distribution of $\emptyset$ and then with a higher order distribution, commonly known as HCT model in plate bending[19] analysis, with $\emptyset$ and its two derivatives $\emptyset_x$ and $\emptyset_y$ at each nodes as field variables. It may be noted that the linear and higher order interpolation laws have their own restrictions. The convergence of linear model is slow whereas the higher compatible model cannot be successfully employed for composite materials. These short comings have lead to the development, in Section 7, of a mixed model having linear interpolation for both warping and stress function. The constitutive equations may be regarded as functionals with displacement and velocity gradients as independent variables. The material symmetry will be restricted to a monoclinic system which has thirteen material coefficients, in general, for generalized Hooke's law and which

results in three coefficients for plane stress torsion problem. The last Section has been devoted for illustrative examples and discussions of results.

5.2 Historical Review:

Torsion of a cylindrical bar by end couple has a long history beginning with Columb and Cauchy[1]. To Saint Venant belongs the credit of bringing the problem of torsion and flexure of beams under a general theory by the inunciation of the 'principle of elastic equivalence of statically equipollent system of loads.' For torsional analysis, he assumed that the state of strain consists of a simple twist combined with warping of the cross section so that the resultant stress at the ends are statically equivalent to a couple about the axis of the prism which will be equal to the applied torque. Later researches by Clebsh and Voigt have resulted in considerable simplification of Saint Venant's analysis by the introduction of assumptions that there is no normal traction across any plane normal to the axis of prism and the state of stress is independent of the longitudinal coordinate along the prism. Based on the above considerations, the small deflection theory of elastic torsion can be derived easily[1]. Adopting the cartesian coordinates X^1, X^2 and X^3 such that X^1 and X^2 are in the plane of the cross section and X^3 is parallel to the longitudinal axis of the prism and normal to the cross section (Fig. 5-1), the three displacements according to Saint Venant's

and the boundary conditions,

$$\phi = \phi_{Bi} \quad \text{(Constants)} \tag{5.2-7}$$

On the boundary Bi (i = 1, ..., N) where Bi are simple non-intersecting boundaries of N fold connected region and G is the shear modulus of the material. From Eq. (5.2-5) the applied torque T can be shown to be,

$$T = 2 \int \int \phi \, d X^1 \, dX^2 \tag{5.2-8}$$

the integration being extended over the cross-section. It is well known that stress function ϕ is similar to the deflection of a membrane. This membrane analogy provide a convenient and intuitive vizualization of the torsion stress function.

With either of the two approaches outlined above, the problem of torsion is reduced to standard problem in the theory of potentials in two dimensions. The most powerful tool for the analysis of potential problems comes from complex variable and associated approach which has been developed in the monumental work by Muskhelishvili[2] and a shorter account by Sokolnikoff[3].

For the case of anisotropic material, it has been shown that a simple affine transformation reduces the problem similar to an isotropic case but with a different cross-section of the prism[1,4].

The problem of stress distribution in a plastic cross section of a prismatic bar subjected to torsion is an internally statically determinate problem. In the case of homogeneous isotropic rigid perfectly plastic material, Nadai[5] showed that the stress function $\emptyset$ must satisfy,

$$| \text{grad } \emptyset | = k \tag{5.2-9}$$

within a simply connected cross section, where k is the limit stress of the material together with Eq. (5.2-7) on the boundaries (where $i = 1$) and thus proposed his sand-hill analogy. For elastoplastic solution, combined membrane and sand-hill analogy may be employed. In 1941, Sadowskey[6] extended the sand-hill analogy for multiply connected cross section. But as discussed by Prager and Hodge[7], the combined membrane and sand-hill analogy is valid only if the plastic region remains in the domain of influence of the boundary curves. Hodge[8] has shown the plastic unloading (where stresses do not increase monotonically with angle of twist) aspect of multiply connected cross sections. This phenomena still requires further investigation to findout the detailed criteria for the unloading.

Elasto-plastic stress analysis presents formidable mathematical difficulties even in very simple cases. Plastic zone may be evaluated by slip line[9], but the elastic zone stress defies almost any kind of theoretical approach owing to the complex shape of the zone boundary. For this reason,

practically all known examples use inverse methods, where the boundary is derived from an assumed solution and an assumed elastoplastic separation line. Essentially the same indirect method has been discussed by Sokolovsky[10] and Zener[11]. Due to absence of any direct method, an approximation formula for warping under large twist has been given by Hodge[12].

Due to these complexities, the Saint Venant torsion problem acquires each year several new solutions by known methods. The conformal representation of assignable shapes has less significance, because of the diversified nature of the practical problems and relative ease with which torsion problems can be solved, in the practical sense, by finite difference or finite element method. Because of its versatility and since irregular shapes, constitutive relations and material nonhomogeity do not pose any further difficulty, finite element method is more preferable.

Herrmann[20] first applied finite element method for the torsion analysis of elastic prism with linear warping function over a triangular element. Shah et al.[21] utilised the minimum complementary energy principle for the generation of finite element equation for nonlinear material with linear distribution of stress function over a triangular element.

5.3 Completeness of Prandtl's Stress Function:

Consider a material body of N-fold connected region and its interior and boundaries denoted by D and Bi where i = 1, 2, ..., N and each Bi represents a simple nonintersecting surface. In the absence of body forces and couple stresses, the Cauchy's laws of motion, i.e., the stress equations of equilibrium for the continuous medium reduces to,

$$T_{mk,m} = 0; \quad T_{mk} = T_{km} \qquad (5.3\text{-}1)$$

where $T_{mk} \in C^2$ is symmetric stress tensor and referred to rectangular cartesian coordinate system. A general solution of (5.3-1) is a representation of T_{mk} with sufficient functional arbitrariness to span the class of all $T_{km} \in C^2$ in D that satisfy (5-3.1). The first stress function solution of (5.3-1) was given by Airy for two dimensional case and Maxwell (1870) and Morera (1892) for three dimensional case. Beltrami's generalization[13,14], in tensorial form, may be read as,

$$T_{km} = \epsilon_{kip}\,\epsilon_{mjq}\,A_{ij,pq}\ ; \quad A_{ij} = A_{ji} \qquad (5.3\text{-}2)$$

Completeness proofs for Beltrami's stress solution were given by many authors, e.g., Dorn and Schild[15] which are all valid for simply connected region. Rieder[16] has shown that representation is incomplete and can atmost provide solutions of (5.2-1) which represent totally self equilibrated force system on each surface Bi. Gurtin[17] is the first to establish

a complete representation in terms of another symmetric second-order tensor field and a vector field, which admits the solutions of (5.3-1) and can be written as,

$$T_{km} = \epsilon_{kip}\,\epsilon_{mjq}\,A_{ij,pq} + \nabla^2(H_{k,m} + H_{m,k}) - H_{j,jkm}$$

$$A_{ij} = A_{ji}\ ;\quad \nabla^4 H_k = 0 \tag{5.3-3}$$

He has proved that this solution is complete in the sense that every (not necessarily totally self equilibrated) solution of (5.3-1) may be represented in the form (5.3-3) for a N-fold connected region. The main aim of this section is to show that atleast for the Saint-Venant's classical torsion problem Gurtin's representation leads to Prandtl's stress function.

From the classical Saint Venant assumption, the only nonvanishing shear stress components for a torsion problem are such that,

$$T_{31} = T_{31}(X^1, X^2) \text{ and } T_{32} = T_{32}(X^1, X^2) \tag{5.3-4}$$

consequently the stress functions A_{ij} and H_j in (5.3-3) will also be such that,

$$A_{ij} = A_{ij}(X^1, X^2) \text{ and } H_j = H_j(X^1, X^2) \tag{5.3-5}$$

Expanding (5.3-3) for T_{31} and T_{32} in (5.3-4) with consideration of (5.3-5) will result in,

$$T_{31} = \Phi_{,2} + \Psi_{,1}\ ,\quad T_{32} = -\Phi_{,1} + \Psi_{,2}$$

$$\text{and}\quad \Psi_{,11} + \Psi_{22} = 0 \tag{5.3-6}$$

with the boundary condition (on the assumption that the peripheral surface is stress free),

$$T_n = \frac{d\Phi}{ds} + \frac{d\Psi}{dn} = 0 \qquad (5.3\text{-}7)$$

where

$$\Phi = \Delta_{23,1} - \Delta_{13,2}$$
$$\Psi = H_{3,11} - H_{3,22} \qquad (5.3\text{-}8)$$

T_n is the resultant fraction normal to the peripheral boundary surface and n and s are respectively the outward normal and tangential directions on the boundary surface at a point.

Now since Ψ is a plane harmonic function, which is apparent from Eq. $(5.3\text{-}6)_3$, there exists a conjugate function Λ which is such that $\Psi + i\Lambda$ is a function of the complex variable $X^1 + iX^2$. Hence from the Cauchy-Riemann relation, if Λ is found, Ψ can be evaluated by means of the equations,

$$\Psi_{,1} = \Lambda_{,2} \text{ and } \Psi_{,2} = -\Lambda_{,1} \qquad (5.3\text{-}9)$$

Substituting (5.3-9) in (5.3-6),

$$T_{31} = (\Phi + \Lambda)_{,2} \text{ and } T_{32} = -(\Phi + \Lambda)_{,1} \qquad (5.3\text{-}10)$$

while $(5.3\text{-}6)_3$ will be automatically satisfied. The boundary condition (5.3-7) will yield,

$$T_n = \frac{d}{ds}(\Phi + \Lambda) = 0 \qquad (5.3\text{-}11)$$

In (5.3-10) and (5.3-11), if $\Phi + \Lambda$ is replaced by $\tilde{\Phi}$, then it shows that $\tilde{\Phi}$ is nothing but the stress function put

forward by Prandtl. Further, it is easy to show that,

$$\Lambda_{,11} + \Lambda_{,22} = 0 \qquad (5.3\text{-}12)$$

Since Prandtl's stress function has to satisfy a Poisson's equation as compatibility criteria, it is clear from (5.3-12) that Λ will provide the homogeneous part of the solution, while $\Phi + \Lambda$ will supply the general solution.

5.4 Finite Element Derivation From Minimum Complementary Energy:

In this Section, finite element equations will be derived through minimization of complementary energy in terms of nodal field variables for a triangular finite element model. It will be assumed that the stress function ϕ within an element is expressible in terms of the nodal field variables through some interpolation law. The nodal field variables and the interpolation law will be left unspecified in this section and two special cases will be derived in two subsequent sections.

Similar to the cases of previous derivations, let the cross section of the member be discretised into R number of triangular elements having N number of nodal points. A typical nodal point may be denoted by p and an element by m. A particular node p will be common to M number of surrounding elements. For an element m, the nodes are designated by 1, 2 and 3, their coordinates by (X_1^1, X_1^2), (X_2^1, X_2^2) and (X_3^1, X_3^2)

(Fig. 5-2). All quantities which are asterisked represent nondimentional form and their relations with dimensional quantities are given in Table (5-1), where 'a' is a typical dimension in the cross section. Stress function at any point within an element 'm' can be expressed in terms of the field variables (nodal unknowns belonging to the element) by the relation,

$$\phi^* = \langle \zeta \rangle \{b\} \tag{5.4-1}$$

where, $\langle \zeta \rangle$ is the interpolation function and $\{b\}$ is the chosen field variable vector appropriate for the element m. Differentiating the equation (5.4-1) with respect to X^1 and X^2, the stress matrix can be written down from Eq. (5.2-5). Thus,

$$\{\overset{*}{\sigma}\} = \frac{1}{\tau_o} \begin{Bmatrix} T_{13} \\ T_{23} \end{Bmatrix} = |S| \{b\} \tag{5.4-2}$$

The constitutive equations relating the shear stress vs. shear strain is assumed to be monotonically increasing function and each volume element will be subjected to proportional loading so that Hencky's total strain energy theory is applicable[18]. The relation between resultant shear stress and shear strain can be expressed as,

$$\gamma = \gamma_o f(\tau/\tau_o) \tag{5.4-3}$$

where $\gamma^2 = (E_{13})^2 + (E_{23})^2$ and $\tau^2 = (T_{13})^2 + (T_{23})^2$

and f is a single valued continuous function, and γ_o and τ_o may assume any value required for curve fitting and non-

dimensionalisation. From Hencky's total strain energy theory,

$$T_{13}/E_{13} = T_{23}/T_{23} \qquad (5.4\text{-}4)$$

From Eqs. (5.4-3) and (5.4-4), the strain expressions are given by,

$$E_{13} = \gamma_o\, \emptyset,_2 . f\,(\tau^*)/\tau_o\, \tau^* \qquad (5.4\text{-}5)$$

and $E_{23} = -\gamma_o\, \emptyset,_1 . f(\tau^*)/\tau_o\, \tau^*$

where, $\tau^* = \tau/\tau_o$

The corresponding matrix relation in terms of nodal field variable $\{b\}$ is

$$\{\epsilon\} = \begin{Bmatrix} E_{13} \\ E_{23} \end{Bmatrix} = \frac{\gamma_o\; f(\tau^*)}{\tau^*}\, |S|\,\{b\} \qquad (5.4\text{-}6)$$

By definition, the complementary energy density for the element 'm' becomes,

$$\Delta U_{co} = \tau_o \int_0^{\tau^*} \{\epsilon\}^T\, d\{\sigma\}^* \qquad (5.4\text{-}7)$$

Substituting Eqs. (5.4-2) and (5.4-6) into (5.4-7) and noting that the matrix $|S|$ is invariant with respect to stresses and deformations, the complementary energy expression reduces to,

$$\Delta U_{co} = \tau_o\, \gamma_o \int_0^{\tau^*} \left(\frac{f(\tau^*)}{\tau^*}\, \{b\}^T\, |C|\, d\{b\} \right) \qquad (5.4\text{-}8)$$

where $|C| = |S|^T\, |S|$ (5.4-9)

Using the resultant stress and strain relation, and Eq. (5.4-2),

it can be easily shown,

$$d\tau^* = \{b\}^T |C| \, d\{b\} / \tau^* \qquad (5.4\text{-}10)$$

Hence, the total complementary energy for the m-th element ΔU_{co} can be obtained from Eq. (5.4-8) using (5.4-10). Thus,

$$\Delta \underline{U}_{co} = \tau_o \, \gamma_o \, La^2 \iint_{A_m} \left| \int_0^{*} f(\tau^*) \, d\tau^* \right| dx^* \, dy^* \qquad (5.4\text{-}11)$$

where A_m is the nondimensional area of the element and L is the length of the prismatic bar. The complementary work is given by,

$$\underline{W} = 2L\theta \, a^2 \iint_{A_m^*} \phi \, dx^* \, dy^* = 2L\theta \, \tau_o a^3 \langle Z \rangle \{b\} \qquad (5.4\text{-}12)$$

where $$\langle Z \rangle = \iint_{A_m^*} \langle \zeta \rangle \, dx^* \, dy^* \qquad (5.4\text{-}13)$$

Therefore, the complementary potential energy π for the overall member can be obtained from (5.4-11) and (5.4-12) after summing over all the R elements. Hence,

$$\pi = \tau_o \, \gamma_o \, La^2 \sum_{m=1}^{R} \left(\iint_{A_m^*} \left| \int_0^{\tau^*} f(\tau^*) \, d\tau^* \right| dx^* \, dy^* \right.$$
$$\left. -2\theta^* \langle Z \rangle \{b\} \right)$$

Differentiating π with respect to the field variables for each node from 1 to N and equating each result to zero, N number of matrix equations will be obtained in N number of unknown set of field variables.

After differentiation, the finite element equation for the typical node p will be of the form,

$$\Sigma' \, |\tilde{G}| \quad \{b\} \quad = \quad 2\theta^* \quad \Sigma' \{Z\} \tag{5.4-14}$$

where,

$$|\tilde{G}| = \iint_{A_m^*} |C| \, \frac{f(\tau^*)}{\tau^*} \; dx^* \, dy^* \tag{5.4-15}$$

and Σ' is the summation extending the M elements which have the common node point p.

It may be noted that for linearly elastic case $f(\tau^*)/\tau^*$ assumes the constant value which is the inverse of shear modulus G and for this the resulting equations leads to N number of linear algebraic matrix equations which can be solved directly. However, for the nonlinear form of $f(\tau^*)/\tau^*$ standard step by step method can be adopted for the solution to be marched out from the unstressed state to a required value of the angle of twist.

5.5 Linear Model:

For linear model, each node has only one degree of freedom and the chosen field variable $\{b\}$ is,

$$\{b\} = \begin{Bmatrix} \phi_1^* \\ \phi_2^* \\ \phi_3^* \end{Bmatrix} \tag{5.5-1}$$

The interpolation function $\langle \zeta \rangle$ can be derived to,

$$\langle \zeta \rangle \quad = \quad \langle \bar{\zeta} \rangle \quad |g| \tag{5.5-2}$$

where $$\langle \zeta \rangle \quad = \quad \langle \; 1 \quad x^* - x_1^* \quad y^* - y_1^* \; \rangle \tag{5.5-3}$$

$$|g| = \frac{1}{2A} \begin{bmatrix} 1 & 0 & 0 \\ y_2^* - y_3^* & y_3^* - y_1^* & y_1^* - y_2^* \\ x_3^* - x_2^* & x_1^* - x_3^* & x_2^* - x_1^* \end{bmatrix} \quad (5.5\text{-}4)$$

and $\quad 2A = (x_2^* - x_1^*)\,(y_3^* - y_1^*) - (x_3^* - x_1^*)\,(y_2^* - y_1^*) \quad (5.5\text{-}5)$

The matrices $|S|$ and $|C|$ are respectively given by,

$$|S| = \begin{bmatrix} g_{31} & g_{32} & g_{33} \\ -g_{21} & -g_{22} & -g_{23} \end{bmatrix} \quad (5.5\text{-}6)$$

and

$$|C| = \begin{bmatrix} g_{21}^2 + g_{31}^2 & g_{21}g_{22} + g_{31}g_{32} & g_{21}g_{23} + g_{31}g_{33} \\ & g_{22}^2 + g_{32}^2 & g_{22}g_{23} + g_{32}g_{33} \\ \text{Symm.} & & g_{23}^2 + g_{33}^2 \end{bmatrix} \quad (5.5\text{-}7)$$

The matrix $\langle Z \rangle$ can be evaluated from Eq. (5.4-13) by using (5.5-2),

$$\langle Z \rangle = \langle Z_1 \quad Z_2 \quad Z_3 \rangle \quad (5.5\text{-}8)$$

where
$$\begin{aligned} Z_1 &= I_x\, g_{21} + I_y\, g_{31} + A \\ Z_2 &= I_x\, g_{22} + I_y\, g_{32} \\ Z_3 &= I_x\, g_{23} + I_y\, g_{33} \end{aligned} \quad (5.5\text{-}9)$$

and,

$$I_x = \iint_{A_m^*} (x^* - x_1^*)\, dx^*\, dy^*$$

$$I_y = \iint_{A_m^*} (y^* - y_1^*)\, dx^*\, dy^*$$

Substituting these expression in Eq. (5.4-14) and differentiating with respect to ϕ_p where ϕ_p is the value of the stress function at the typical node p, the required equations can be obtained.

The principle advantage of linear model is that it is very simple and requires very limited memory space in computer. But the disadvantage is that, though it gives monotonic convergence, the rate of convergence is rather poor. However, some acceleration scheme, such Atkin's del-square or Richardson's extrapolation[22] may be applied to improve the convergency of the solution.

5.6 Refined Model:

In this model, the stress function ϕ as well as its slopes ϕ_x and ϕ_y will also be continuous over the edges of an element. It is somewhat awkward to achieve slope compatibility for a triangular model; however this result has been achieved by dividing the element into three subelements. This procedure for deriving a slope compatible model was first developed by Clough and Tocher[19] for plate bending analysis and is known as HCT model. Steps leading to the relation corresponding to Eq. (5.5-2) are as follows.

1) Each triangle can be subdivided into three subelements by choosing a point '0' and joining the vertices. The coordinate of '0' may be any where within the element but for convenience it may be chosen as the center of gravity of the triangle (x_o^*, y_o^*).

2. Transform the global coordinate (x^*, y^*) to (x', y') by shifting the origin to '0' as shown in Fig. (5-3).

3. The following steps have to be repeated for each subelement 'l' (l = 1, 2 and 3) having nodes (m, n, 0) as in Fig. (5-4).

(a) Transform the coordinates from (x', y') system to local system $(\bar{x}, \bar{y})$ such that $\bar{x}$ and $\bar{y}$ axes are respectively parallel and normal to the side mn. The transformation will be given by,

$$\begin{aligned} \bar{x} &= x' \cos \theta' - y' \sin \theta' \\ \bar{y} &= x' \sin \theta' + y' \cos \theta' \end{aligned} \qquad (5.6\text{-}1)$$

where $\tan \theta' = (y_n - y_m) / (x_m - x_n)$

(b) Assume stress function ϕ^* over the subelement l as,

$$\phi^* = \langle \bar{\zeta} \rangle \quad \alpha^l \qquad (5.6\text{-}2)$$

where $\langle \bar{\zeta} \rangle = \langle 1 \ \bar{x} \ \bar{y} \ \bar{x}^2 \ \overline{xy} \ \bar{y}^2 \ \bar{x}^3 \ \overline{xy}^2 \ \bar{y}^3 \rangle$ (5.6-3)

and $\langle \alpha^l \rangle = \langle \alpha_1 \ \alpha_2 \ \ldots \ \alpha_9 \rangle^l$

Then the local field variables may be given as,

$$\{\bar{b}\} = \begin{Bmatrix} \phi^* \\ \phi^*_{,\bar{x}} \\ \phi^*_{,\bar{y}} \end{Bmatrix} = \begin{bmatrix} 1 & \bar{x} & \bar{y} & \bar{x}^2 & \overline{xy} & \bar{y}^2 & \bar{x}^3 & \overline{xy}^2 & \bar{y}^3 \\ 0 & 1 & 0 & 2\bar{x} & \bar{y} & 0 & 3\bar{x}^2 & \bar{y}^2 & 0 \\ 0 & 0 & 1 & 0 & \bar{x} & 2\bar{y} & 0 & 2\overline{xy} & 3\bar{y}^2 \end{bmatrix} \{\alpha^l\} \qquad (5.6\text{-}4)$$

It is to be noted here that the stress function and its slopes will be uniquely defined along the edge mn if the field variables $\{\bar{b}\}$ is prescribed only at the nodes m and n and

this ensures the slope compatibility. Now, calculate $\{\bar{b}\}$ at the five points m, n, O, p and q where p and q are the mid points of Or and On.

c) Transform $\{\bar{b}\}$ to the global system $\{b\}$ by the transformation,

$$\{b\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta' & \sin\theta' \\ 0 & -\sin\theta' & \cos\theta' \end{bmatrix} \{\bar{b}\} \quad (5.6\text{-}5)$$

d) At p and q normal slopes will be required. Denoting p or q by r, the normal slope at r can be obtained as,

$$\phi^{*}{}_{,\bar{n}_r} = (\phi^{*}_{,x'} \sin\theta'' + \phi^{*}_{,y'} \cos\theta'')_{\text{at } r} \quad r = p, q \quad (5.6\text{-}6)$$

$$\text{where} \quad \theta''|_{\text{at } r} = \tan^{-1}(-y'/x')\,|_{\text{at } r} \quad (5.6\text{-}7)$$

and $\bar{n}_r$ is normal to the corresponding side at r. Hence, five vectors have been obtained for each subelement, e.g., $\underset{\sim}{b}^1_m$, $\underset{\sim}{b}^1_n$, $\underset{\sim}{b}^1_o$, $\underset{\sim}{b}^1_p$ and b^1_q. The first three has three elements in each, while each of the last two has one element. Keeping this in mind, in general, the relation can be written, in the form,

$$\{\underset{\sim}{b}^1 k\} = \phi^1_k \{\alpha^1\} \qquad \text{(no summation over 1)} \quad (5.6\text{-}8)$$

where 1 = 1, 2 and 3; k = m, n, o, p and q.

4) The vectors $\underset{\sim}{b}^1{}_k$ are not all independent with respect to the overall element, because they have to satisfy inter-subelement compatibility which can be written in the following manner,

$$\left\{ \frac{\tilde{b}}{\tilde{0}} \right\} = \begin{bmatrix} A_{11} & A_{12} \\ & \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{Bmatrix} \qquad (5.6\text{-}9)$$

where

$$| A_{11} | = \begin{Bmatrix} \phi_m \\ 0 \\ 0 \end{Bmatrix} \qquad | A_{12} | = \begin{bmatrix} 0 & 0 \\ \phi^2_m & 0 \\ 0 & \phi^3_m \end{bmatrix}$$

$$| A_{21} | = \begin{Bmatrix} \phi^1_n \\ 0 \\ -\phi^1_m \\ \phi^1_0 \\ 0 \\ \phi^1_q \\ 0 \\ -\phi^1_p \end{Bmatrix} \qquad | A_{22} | = \begin{bmatrix} -\phi^2_m & 0 \\ \phi^2_n & -\phi^3_m \\ 0 & \phi^3_n \\ -\phi^2_0 & 0 \\ \phi^2_0 & -\phi^3_0 \\ -\phi^2_p & 0 \\ \phi^2_q & \phi^3_p \\ 0 & \phi^3_q \end{bmatrix}$$

$$(5.6\text{-}10)$$

$$\{ \alpha^1 \} = \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_9 \end{Bmatrix}^1 \qquad (5.6\text{-}11)$$

and

$$\{ \tilde{\underset{\sim}{b}} \} = \begin{Bmatrix} \underset{\sim}{b}_1 \\ \underset{\sim}{b}_2 \\ \underset{\sim}{b}_3 \end{Bmatrix} \qquad (5.6\text{-}12)$$

that is $\{ \tilde{\underset{\sim}{b}} \}$ represents the nodal vector of the field variables for the element, $\underset{\sim}{b}_i$ is that for a particular node i (i = 1, 2 and 3) and $\tilde{0}$ is a null vector. Performing some manipulation for reduction, it can be easily shown,

$$\begin{Bmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{Bmatrix} = | H | \quad \{\tilde{b}\} \tag{5.6-13}$$

where,

$$| H | = \begin{bmatrix} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \\ -A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21}) \end{bmatrix} \tag{5.6-14}$$

Having obtained the relation (5.6-13), the general procedure outlined in Section 3 may be followed to establish the finite element equations. The matrix $|S|$ may now be obtained from (5.6-2) such that,

$$|S| = |\overline{S}| \; | H | \; \{\tilde{b}\} \tag{5.6-15}$$

where,

$$| \overline{S} | = \begin{bmatrix} -0 & -1 & 0 & -2\overline{x} & -\overline{y} & 0 & 3\overline{x}^2 & -\overline{y}^2 & 0 \\ 0 & 0 & 1 & 0 & \overline{x} & 2\overline{y} & 0 & 2\overline{xy} & 3\overline{y}^2 \end{bmatrix} \tag{5.6-16}$$

Finally the matrix will be given by,

$$< \zeta > = < \overline{\zeta} > | H | \tag{5.6-17}$$

where $< \overline{\zeta} >$ is defined in (5.6-3). The main disadvantage of this model is the complicated expressions for integration to obtain $| G |$ and $< Z >$. The best way to evaluate these matrices is to apply numerical integration technique over each subelement. However, if it is assumed that the elements are quite small in size and $f(\tau^*)/\tau^*$ will be practically constant over an element, the computational economy can be immensely increased even at the expense of increase number of elements.

5.7 Continuum Approach:

In this section, torsional formulation will be obtained for prismatic members of composite viscous solids having arbitrary cross section. The material properties may vary across one or more surfaces as in reinforced concrete. In the previous formulation with linear stress function, this problem can be solved with minor modifications, but in the neighbourhood region of an interface between any two materials, the convergence and reliability may be too poor. In the 2nd formulation, due to continuity of slopes across the edges of an element, the model will not represent an actual distribution even at the limiting case of infinitesimal element size, where the element edge is also an interface of different materials, because in the actual case there will be jumps in stresses across the boundary. Motivated due to these trammels, a third model has been developed where the warping function as well as stress function has been chosen linearly inside each element and the equations (1.10-1) and (1.10-2) in Chapter 1, have been applied for the derivation of finite element equations.

Assuming the angle of twist θ is a given quantity in cartesian coordinate system (X^1, X^2, X^3), the displacements are,

$$
\begin{aligned}
U_1 &= -\theta X^2 X^3 \\
U_2 &= \theta X^1 X^3 \\
U_3 &= \langle 1 \; X^1 \; X^2 \rangle \; | g | \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}
\end{aligned}
\qquad (5.7\text{-}1)
$$

and the stress function is given by,

$$\emptyset = \langle 1 \; X^1 \; X^2 \rangle \; | g | \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} \tag{5.7-2}$$

where the matrix $| g |$ will be given by,

$$| g | = \frac{1}{2A} \begin{bmatrix} 2A/3 & 2A/3 & 2A/3 \\ X_2^2 - X_3^2 & X_3^2 - X_1^2 & X_1^2 - X_2^2 \\ X_3^1 - X_2^1 & X_1^1 - X_3^1 & X_2^1 - X_1^1 \end{bmatrix}$$

and $\quad 2A = (X_2^1 - X_1^1)(X_3^2 - X_1^2) - (X_3^1 - X_1^1)(X_2^2 - X_1^2)$

w_i, w_j, w_k and Φ_i, Φ_j and Φ_k are respectively the nodal values of U_3 and $\emptyset$.

The stresses, as derived from $\emptyset$, are constants within an element,

$$\tau_1 = T_{13} = - \langle g_{31} \; g_{32} \; g_{33} \rangle \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}$$

$$\tau_2 = T_{23} = \langle g_{21} \; g_{22} \; g_{23} \rangle \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} \tag{5.7-3}$$

The strain components, derived from displacement distributions, are,

$$\gamma_1 = E_{13} = \langle g_{21} \quad g_{22} \quad g_{23} \rangle \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \theta X^2$$

$$\gamma_2 = E_{23} = \langle g_{31} \quad g_{32} \quad g_{33} \rangle \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} + \theta X^1$$

(5.7-4)

In Eqs. (5.7-3) and (5.7-4) g_{ij} are elements of the matrix $| g |$.

Since stresses from Eq. (5.7-3) are constants within an element, the governing equation (1.10-1) reduces to the form,

$$\Sigma \left| \int_S (P_1 - n_3 \tau_1)\, \dot{U}_1 + (P_2 - n_3 \tau_2)\dot{U}_2 + (P_3 - n_1 \tau_1 - n_2 \tau_2)\, \dot{U}_3 \, ds \right| = 0$$

(5.7-5)

where, P_1, P_2 and P_3 are applied forces over the boundary of the element, S in directions corresponding to X^1, X^2 and X^3; n_1, n_2 and n_3 are out ward normals; the integration is to be performed over the boundary of each element and the summation extends over all the elements comprising the assembly. Since the stress function satisfies equilibrium conditions it will be shown later that (5.7-5) is a redundant equation.

Correspondingly, after minor modifications, Eq. (1.10-2) can be written in the form,

$$\Sigma \left| \int_A (\tau_1 - \hat{\tau}_1)\, \mathring{\gamma}_1 + (\tau_2 - \hat{\tau}_2)\, \mathring{\gamma}_2 \, dA \right| = 0 \quad (5.7\text{-}6)$$

where, $\hat{\tau}_1$ and $\hat{\tau}_2$ are stresses derived from the constitutive equations with assumed displacement distribution. The integratio

extends over the cross sectional area of each element and summation extends over all the elements. Again, since $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are arbitrary strain rates and can be varied independently hence (5.7-6) can be degenerated to,

$$\Sigma \left| \int_A (\tau_1 - \hat{\tau}_1) \dot{\gamma}_1 \, dA \right| = 0 \qquad (5.7\text{-}7)$$

and

$$\Sigma \left| \int_A (\tau_2 - \hat{\tau}_2) \dot{\gamma}_2 \, dA \right| = 0 \qquad (5.7\text{-}8)$$

The constitutive equations for the material in an element can be prescribed in the general form,

$$\hat{\tau}_1 = \hat{\hat{\tau}}_1 (\gamma_1, \gamma_2, \dot{\gamma}_1, \dot{\gamma}_2, t^p)$$

$$\tau_2 = \hat{\hat{\tau}}_2 (\gamma_1, \gamma_2, \dot{\gamma}_1, \dot{\gamma}_2, t^p) \qquad (5.7\text{-}9)$$

As usual, they will be restricted by thermodynamical restriction which will not be discussed here. Since, these equations are nonlinear, parametric differentiation technique may be applied for the solution of problem. Applying this over the constitutive equations, (5.7-9) reduces to,

$$\hat{\tau}_1^* = C_{11} \gamma_1^* + C_{12} \gamma_2^* + \tilde{C}_{11} \gamma_1^* + \tilde{C}_{12} \gamma_2^*$$

and $$\hat{\tau}_2^* = C_{21} \gamma_1^* + C_{22} \gamma_2^* + \tilde{C}_{21} \gamma_1^* + \tilde{C}_{22} \gamma_2^* \qquad (5.7\text{-}10)$$

where,

$$C_{ij} = \frac{\partial \hat{\hat{\tau}}_i}{\partial \gamma_j} \quad ; \quad \tilde{C}_{ij} = \frac{\partial \hat{\tau}_i}{\partial \dot{\gamma}_j} \qquad (5.7\text{-}11)$$

$$C_{ij} = C_{ji} \text{ and } \tilde{C}_{ij} = \tilde{C}_{ji}, \quad i, j = 1, 2$$

From Eqn. (5.7-4), the incremental strains are given by,

$$\gamma_1^* = (g_{21} w_1^* + g_{22} w_2^* + g_{23} w_3^*) - \theta^* X^2$$

$$\gamma_2^* = (g_{31} w_1^* + g_{32} w_2^* + g_{33} w_3^*) + \theta^* X^1 \tag{5.7-12}$$

and time derivatives of γ_1 and γ_2 are similar to (5.7-12) where w_i^* and θ^* are to be replaced by $\dot{w}_i$ and $\dot{\theta}$. Now, since at each step Eqs. (5.7-5) (5.7-7) and (5.7-8) are to be satisfied, after application of parametric differentiation they will be reduced to the forms,

$$\Sigma \left| \int_S (P_1^* - n_3 \tau_1^*)\dot{U}_1 + (P_2^* - n_3 \tau_2^*)\dot{U}_2 + (P_3^* - n_1 \tau_1^* - n_2 \tau_2^*)\dot{U}_3 \;\; ds \right| = 0 \tag{5.7-13}$$

$$\Sigma \left| \int_A (\tau_1^* - \hat{\tau}_1^*) \; \mathring{\gamma}_1 \; dA \right| = 0 \tag{5.7-14}$$

and

$$\Sigma \left| \int_A (\tau_2^* - \hat{\tau}_2^*) \; \gamma_2 \; dA \right| = 0 \tag{5.7-15}$$

Substituting in Eqs. (5.7-14) and (5.7-15), relations for τ_i^* from (5.7-3), $\bar{\tau}_i^*$ from (5.7-10) and expressions for $\mathring{\gamma}_i$ similar to (5.7-12), the two equations yield,

$$\Sigma \Big| \int_A \{(g_{31} \; \phi_1^* + g_{32} \; \phi_2^* + g_{33} \; \phi_3^*)$$

$$+C_{11} \; (g_{21} w_1^* + g_{22} w_2^* + g_{23} w_3^* - \theta^* X^2)$$

$$+C_{12} \; (g_{31} w_1^* + g_{32} w_2^* + g_{33} w_3^* + \theta^* X^1)$$

$$+\tilde{C}_{11} \; (g_{21} \dot{w}_1^* + g_{22} \dot{w}_2^* + g_{23} \dot{w}_3^* + \dot{\theta}^* X^2)$$

$$+\tilde{C}_{12} \; (g_{31} \dot{w}_1^* + g_{32} \dot{w}_2^* + g_{33} \dot{w}_3^* + \dot{\theta}^* X^1) \}_X$$

$$x\{ g_{21}\, \dot{w}_1 + g_{22}\, \dot{w}_2 + g_{23}\, \dot{w}_3 - \dot{\theta} X^2 \}\, dA \,| \;=\; 0 \qquad (5.7\text{-}16)$$

and

$$\begin{aligned}
\Sigma| \int_A \{ &(g_{21}\, \Phi_1^* + g_{22}\, \Phi_2^* + g_{33}\, \Phi_3^*) \\
&- C_{21}(g_{21}\, w_1^* + g_{22}\, w_2^* + g_{23}\, w_3^* - \theta^* X^2) \\
&- C_{22}(g_{31}\, w_1^* + g_{32}\, w_2^* + g_{33}\, w_3^* + \theta^* X^1) \\
&- \tilde{C}_{21}(g_{21}\, \dot{w}_1^* + g_{22}\, \dot{w}_2^* + g_{23}\, \dot{w}_3^* - \dot{\theta}^* X^2) \\
&- \tilde{C}_{22}(g_{31}\, \dot{w}_1^* + g_{32}\, \dot{w}_2^* + g_{33}\, \dot{w}_3^* + \dot{\theta}^* X^1) \} \\
&\quad \{ g_{31}\, \dot{w}_1 + g_{32}\, \dot{w}_2 + g_{33}\, \dot{w}_3 + \dot{\theta} X^1 \} dA \,| = 0
\end{aligned}$$

$$(5.7\text{-}17)$$

These equations have to be satisfied for arbitrary values of $\dot{w}_1$, $\dot{w}_2$ and $\dot{w}_3$. Hence, after integration equating the coefficients of $\dot{w}_1$, $\dot{w}_2$ and $\dot{w}_3$ separately to zero will lead to six governing finite element equations corresponding to six unknowns per element. The equations can be written in a matrix form for the node i where, i may be 1, 2 or 3;

$$\Sigma\, A' \left(|\tilde{G}|_i \begin{matrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{matrix} + |G|_i \begin{matrix} w_1 \\ w_2 \\ w_3 \end{matrix} + |H|_i \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} \right) = \Sigma \{ l_i \}$$

$$(5.7\text{-}18)$$

where A' is the area of the element. Each $|\tilde{G}|_i$, $|G|_i$ and $|H|_i$ are 2 x 3 matrices and the elements of the matrices are as follows:

$$G_{1j} = g_{2i} (g_{2j} C_{11} + g_{3j} C_{12})$$

$$G_{2j} = g_{3i} (g_{2j} C_{21} + g_{3j} C_{22})$$

$$H_{1j} = g_{2i} g_{3j} \qquad (5.7\text{-}19)$$

$$H_{2j} = g_{3i} g_{2j} \qquad \text{where } j = 1, 2 \text{ and } 3.$$

$$l_1 = g_{2i} A' \{(+ C_{11}\bar{y} - C_{12}\bar{x}) \theta^* + (\tilde{C}_{11}\bar{y} - \tilde{C}_{12}\bar{x}) \dot{\theta}^*\}$$

$$l_2 = g_{3i} A' \{(- C_{21}\bar{y} + C_{22}\bar{x}) \theta^* + (-\tilde{C}_{21}\bar{y} + \tilde{C}_{22} \bar{x}) \dot{\theta}^* \}$$

$$A'\bar{y} = \int_A X^2 \, dA \quad \text{and} \quad A'\bar{x} = \int_A X^1 \, dA \qquad (5.7\text{-}20)$$

The elements $\tilde{G}_{ij}$ can be obtained from G_{ij} by replacing C_{11}, C_{12} and C_{22} by $\tilde{C}_{11}$, $\tilde{C}_{12}$ and $\tilde{C}_{22}$ respectively.

Considering $\dot{\theta} \neq 0$, the remaining portion, i.e., the coefficients of $\dot{\theta}$ in (5.7-16) and (5.7-17) gives rise to a more concise form,

$$\Sigma \Big| \int_A (\tau_1^* - \tau_1^*) X^2 \, dA = 0$$

$$\Sigma \Big| \int_A (\tau_2^* - \tau_2^*) X^1 \, dA = 0 \qquad (5.7\text{-}21)$$

Consider, now, the Eq. (5.7-13). Since the bar is prismatic, the surface boundary of each element, s, can be subdivided into two parts, s' the vertical surface and s'' the horizontal surface in the plane of cross-section. Hence,

for s', $\quad n_3 = 0$

and for s'', $n_1 = n_2 = 0$ and $n_3 = 1$

Concentrating on the surface s" and integrating over the whole cross-section, (5.7-13) will yield,

$$T^{*} = 2\,\Sigma \left| \int_{A} \Phi^{*} \; dA \right| \qquad (5.7\text{-}22)$$

where T^{*} is the increment in torsional moment on the surface s". Again, isolating the surface s', the equation reduces to,

$$\Sigma \left| \int_{s'} P_3^{*} - n_1 \tau_1^{*} - n_2 \tau_2^{*} \quad \dot{U}_3 \, ds \right| = 0 \qquad (5.7\text{-}23)$$

It may be noted here that P_3^{*} is not actually an applied force and has arisen because of discretization technique in finite element method. However, it can be evaluated from the displacement distributions through constitutive equations (5.7-10). If that is done, it can be easily shown by applying the Gauss theorem for transforming surface integral to volume integral that if equations (5.7-18) are satisfied, Eq. (5.7-23) will be automatically satisfied. In this system since, no known external forces are considered, the forces evaluated from displacements through constitutive equations may be considered as loading. With this consideration and by Eq. (5.7-13), and (5.7-22) it is not difficult to show that (5.7-21) are redundant equations and torsional moment can be calculated by either of the relations,

$$T^{*} = 2\,\Sigma \left| \int_{A} \Phi^{*} \, dA \right| = \Sigma \left| \int_{A} (\tau_2^{*} X^1 - \tau_1^{*} X^2) \; dA \right| \qquad (5.7\text{-}24)$$

The overall equations can be obtained by appropriately superposing Eq. (5.7-18) for the entire assemblage and this will give rise to simultaneous first order differential equations in terms of incremental nodal values of Φ and w. This can be solved by any standard numerical integration scheme.

5.8 Numerical Results:

Fig. (5-5) shows the arrangements of elements in the cross-sections considered for examples. The shear stress-shear strain relations considered for different cases are shown in Fig. (5-6). Therein, the experimental curves 1, 2 and 3 are for circular cross-sections and 4 for rectangular cross-section as reported by Buchanan[23]. The curve 5 is given by the equation,

$$\gamma^* = \sqrt{3} \quad \text{sinhh} \ \tau^* \tag{5.8-1}$$

and the curve 6 is the classical elasto-plastic relation.

The curves C_1, C_2 and C_3 of Fig. (5-7), corresponding to the stress-strain relations 1, 2 and 3 of Fig. (5-6) show the plots of torque vs. twist in non-dimensional form for unit circular cross-sections. These are compared with the curves shown by Buchanan using finite difference method. In Fig. (5-8), the results for rectangular cross-section are obtained using curve 4 of Fig. (5-6). Herein, the finite element solutions are obtained using a linear model.

In Fig. (5-9) plots of T^* vs. θ^* are shown for the cases of unit circular and unit square cross-sections employing linear model and compared with the results of Smith and Sidebottom[18], for the stress strain relation given by Eq. (5.8-1). It is to be noted that the results of Smith and Sidebottom are exact for the circular case, while for the rectangular cross-section energy method is employed. The same problems have been solved using refined model.The values obtained are practically same as given by Smith and Sidebottom. The maximum error for square section is about 0.2 percent whereas for circular section it is about 0.7 percent.

The results for elasto-plastic analysis of circular and rectangular cross-sections using the present analysis are compared with the exact analysis, and are shown in Fig. (5-10).

Fig. (5-11) shows the plots of torque vs. twist for an L-section corresponding to curves 5 and 6 of Fig.(5-6). This has been solved using the linear model only.

Elasto-plastic torsion of a doubly connected square cylinder has been solved and is shown in Fig. (5-12). The dimensions of the cylinder are the same as those given by Stout and Hodge[8] so that the results can be compared. It is seen in Fig. (5-12a) that except at the sharp turning portion, representing the elasto-plastic transformation, the curve

agrees excellently with the result given by Stout and Hodge. However, the maximum discrepancy between values of 2 and 3 for θ^* is not more than 3 percent. But, unfortunately the deviation in stress distribution along the diagonal line as shown in Fig. (5-12b) is quite appreciable. This is because, to save computer time, this diagonal line had been taken as a boundary from symmetry considerations. It has always been observed that error percentage on a boundary line is rather high compared to other inside points. Even so, the nature of the curves are fairly same and unloading aspect from elastic zone is clearly indicated at θ^* equal to 7.06.

It may be noted that analysis of multiply connected region poses no problem, Since at internal boundary contour, the values of ϕ has to be the same everywhere. This is readily accomplished by a modification of the governing matrix equation.

Quantity	Dimensioned	Nondimen-sioned	Remation
Coordinates	$X^1,\ X^2$	x^*, y^*	$X^1=ax^*,\ X^2=ay^*$
Stress function	$\emptyset$	$\emptyset^*$	$\emptyset = {}_o\, a\, \emptyset^*$
Strain		*	$= {}_o{}^*$
Stress		*	$= {}_o{}^*$
Torque	T	T^*	$T = {}_o a^3 T^*$
Angle of twist	θ	θ^*	$\theta = {}_o\theta^*/a$
Area of element 'm'	A_m	A_m^*	$A_m = a^2 A_m^*$

Table 5-1 : Relation between dimensional and nondimensional quantities.

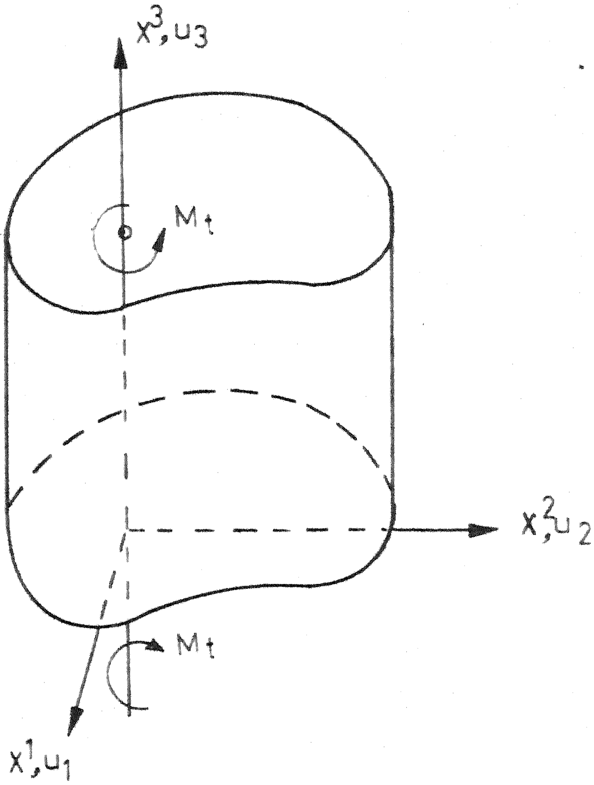


FIG. (5-1) COORDINATE SYSTEM

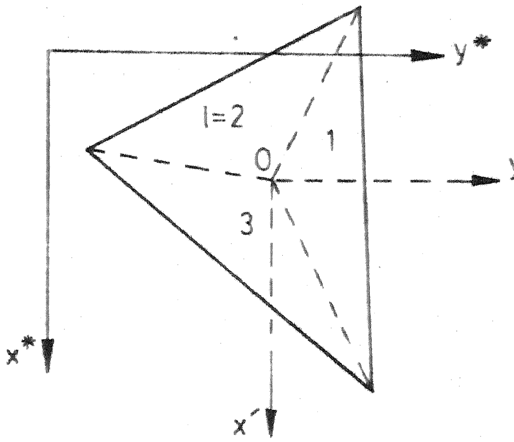


FIG.(5-3) PARALLEL COORDINATE SHIFT

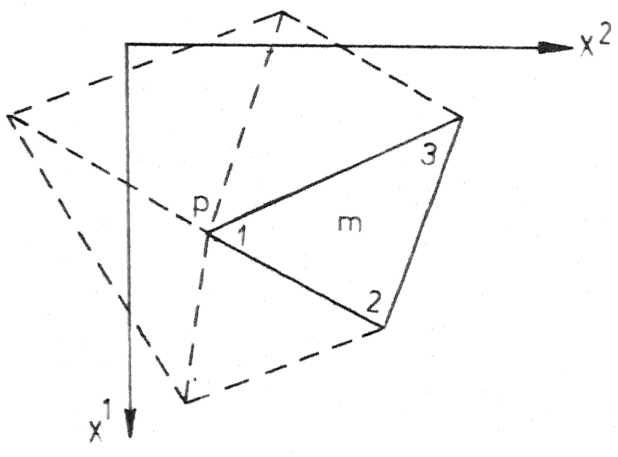


FIG. (5-2) ARRANGMENT OF ELEMENTS

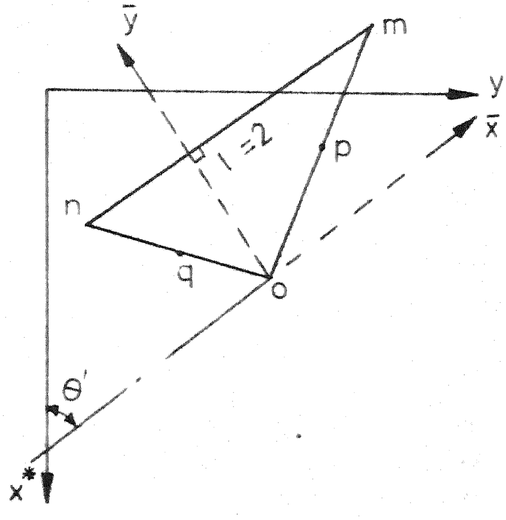


FIG.(5-4) COORDINATE ROTATION

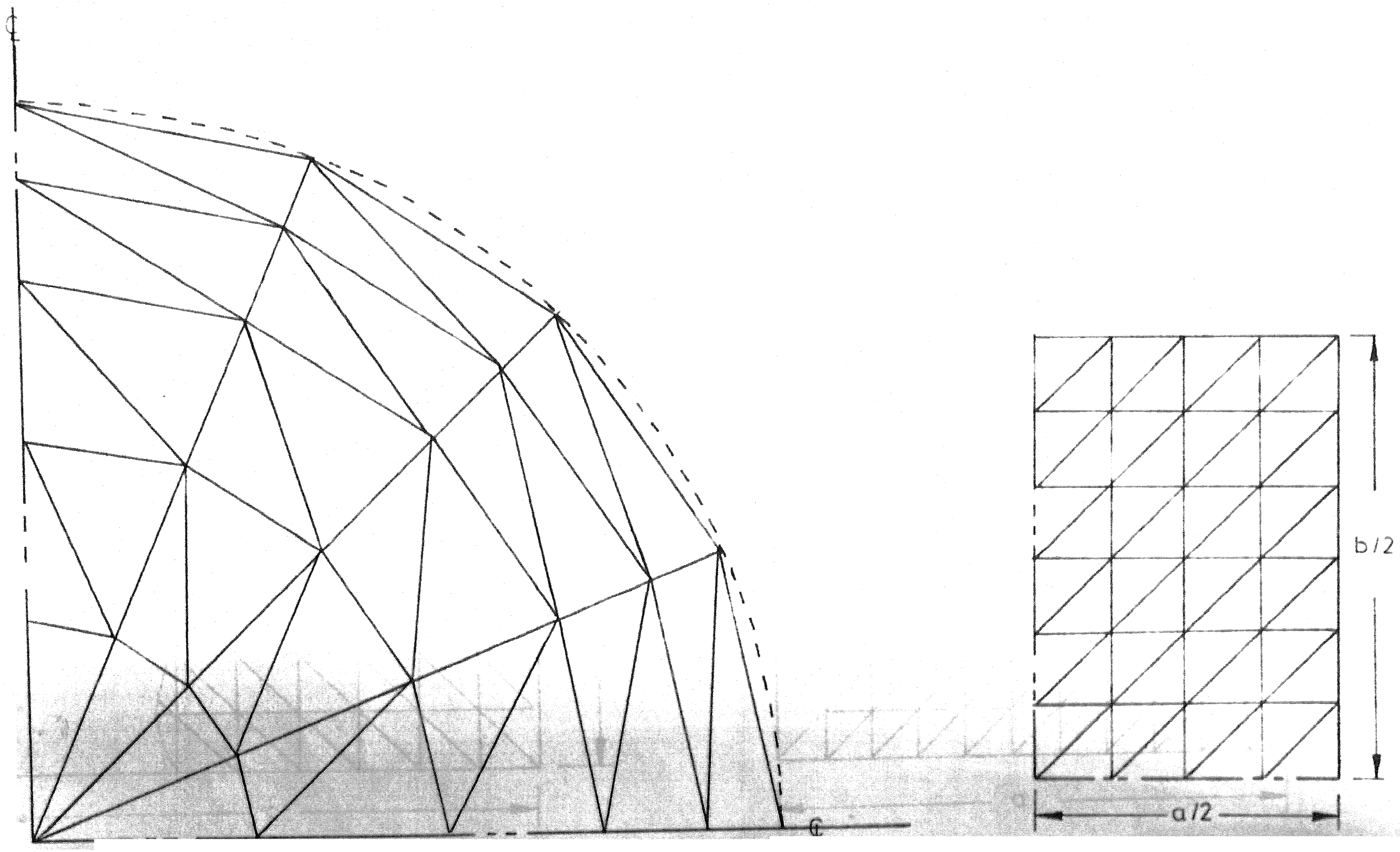


FIG. (5-5a) CIRCULAR SECTION

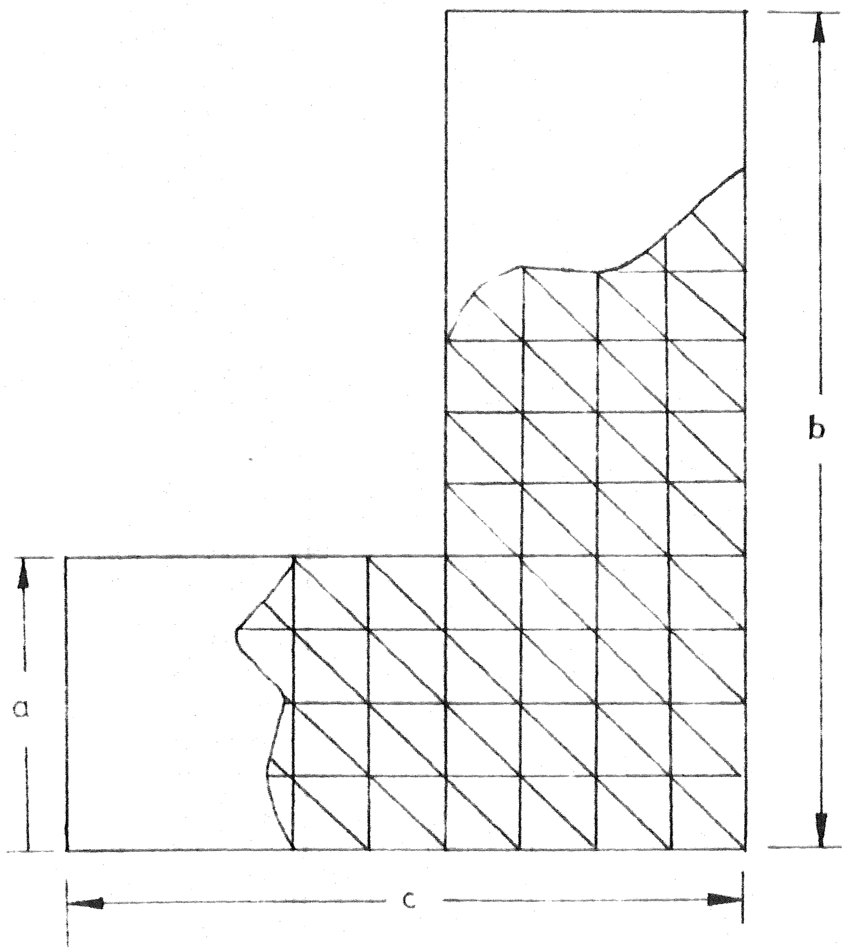


FIG.(5-5c) 'L' SECTION

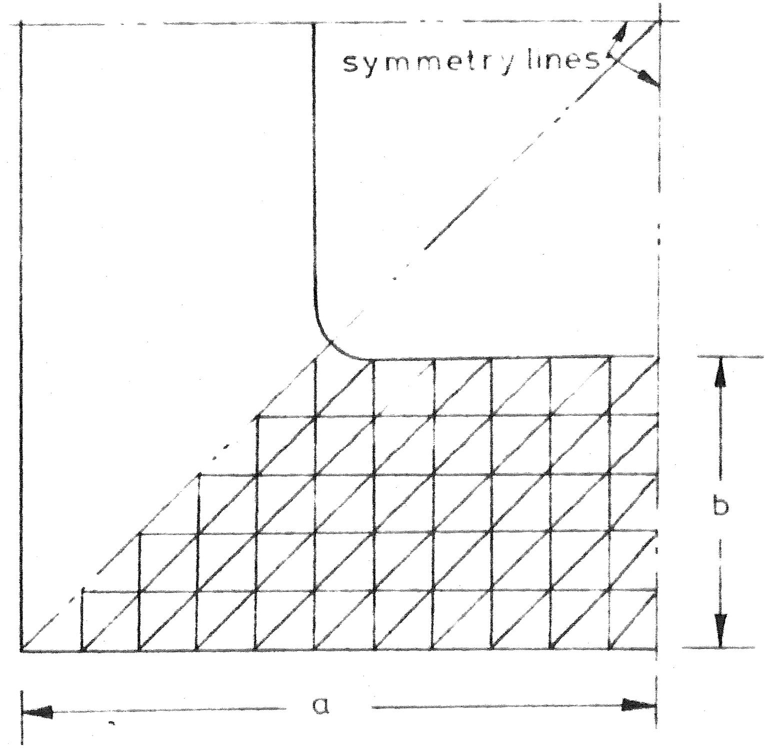


FIG. (5-5d) HOLLOW SQ. CYLINDERICAL SECTION

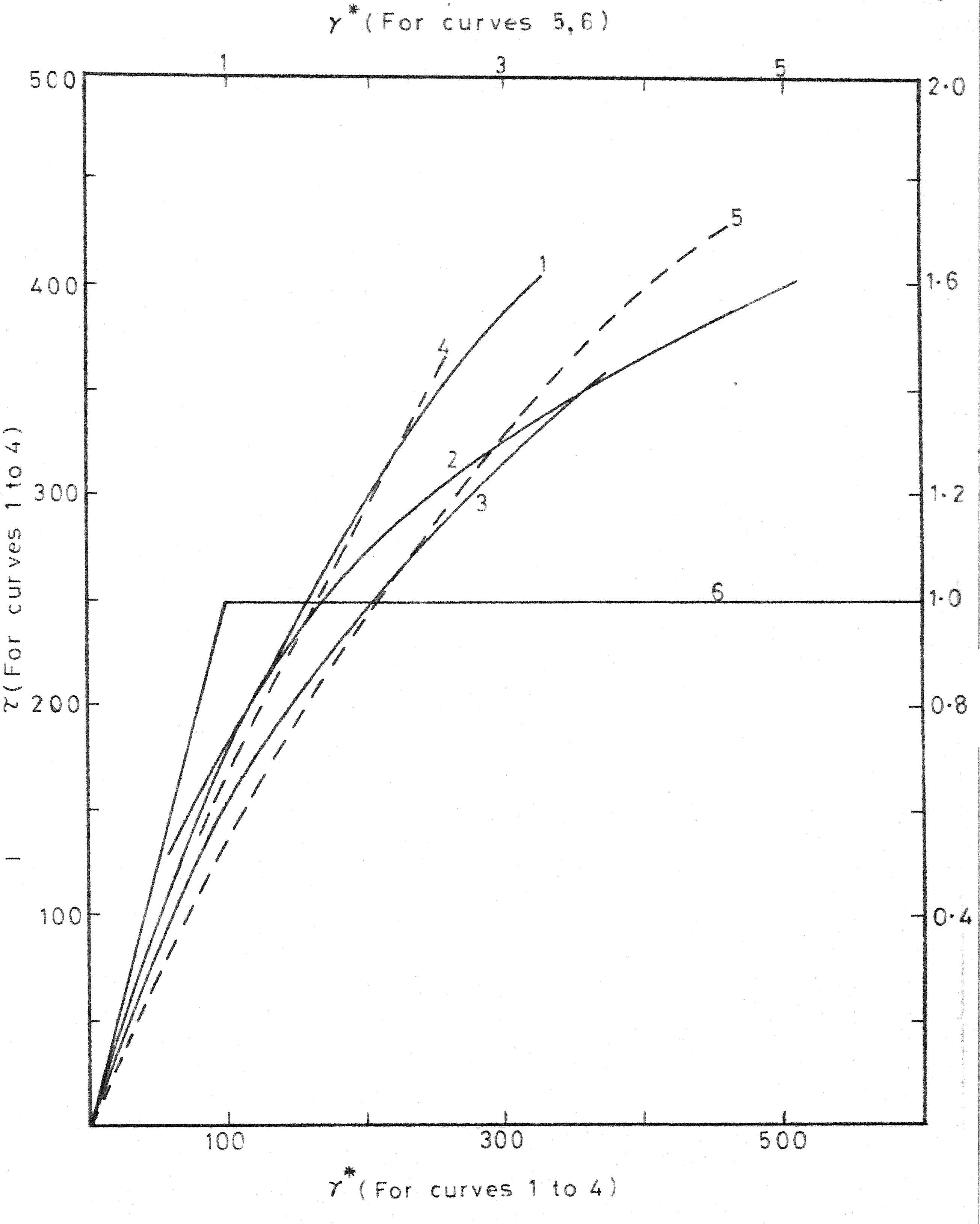


FIG. (5-6) STRESS-STRAIN RELATIONS

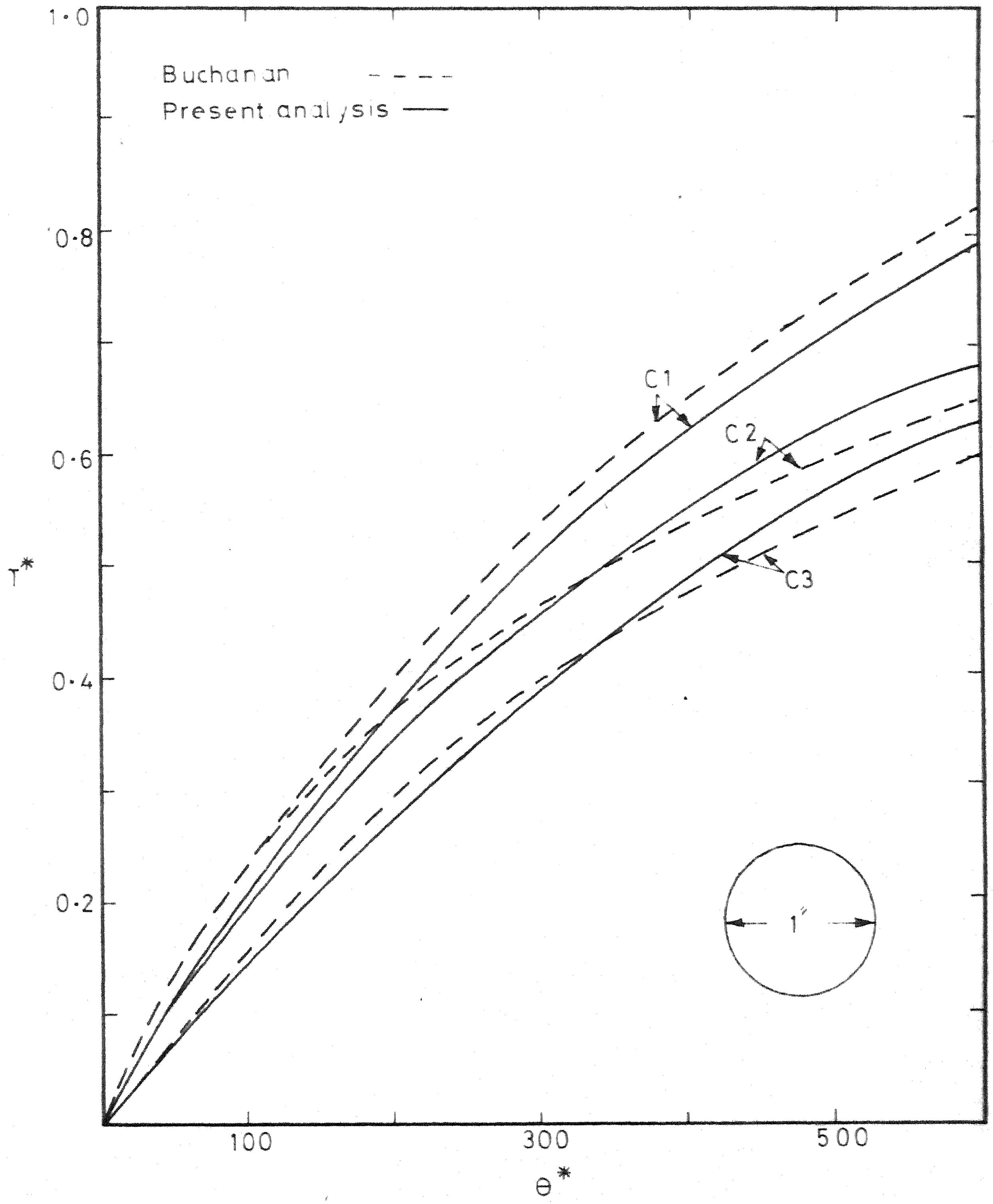


FIG (5-7) T^* Vs θ^* FOR CIRCULAR SECTION

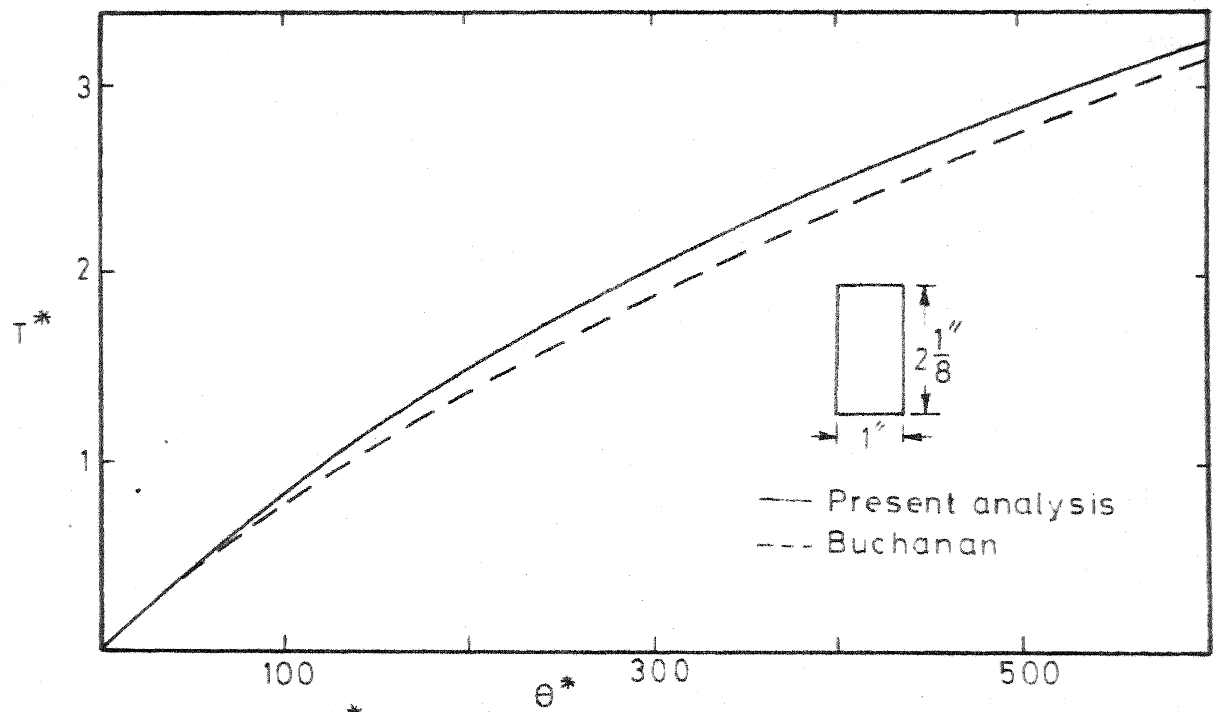


FIG. (5-8) T^* Vs θ^* FOR RECTANGULAR SECTION

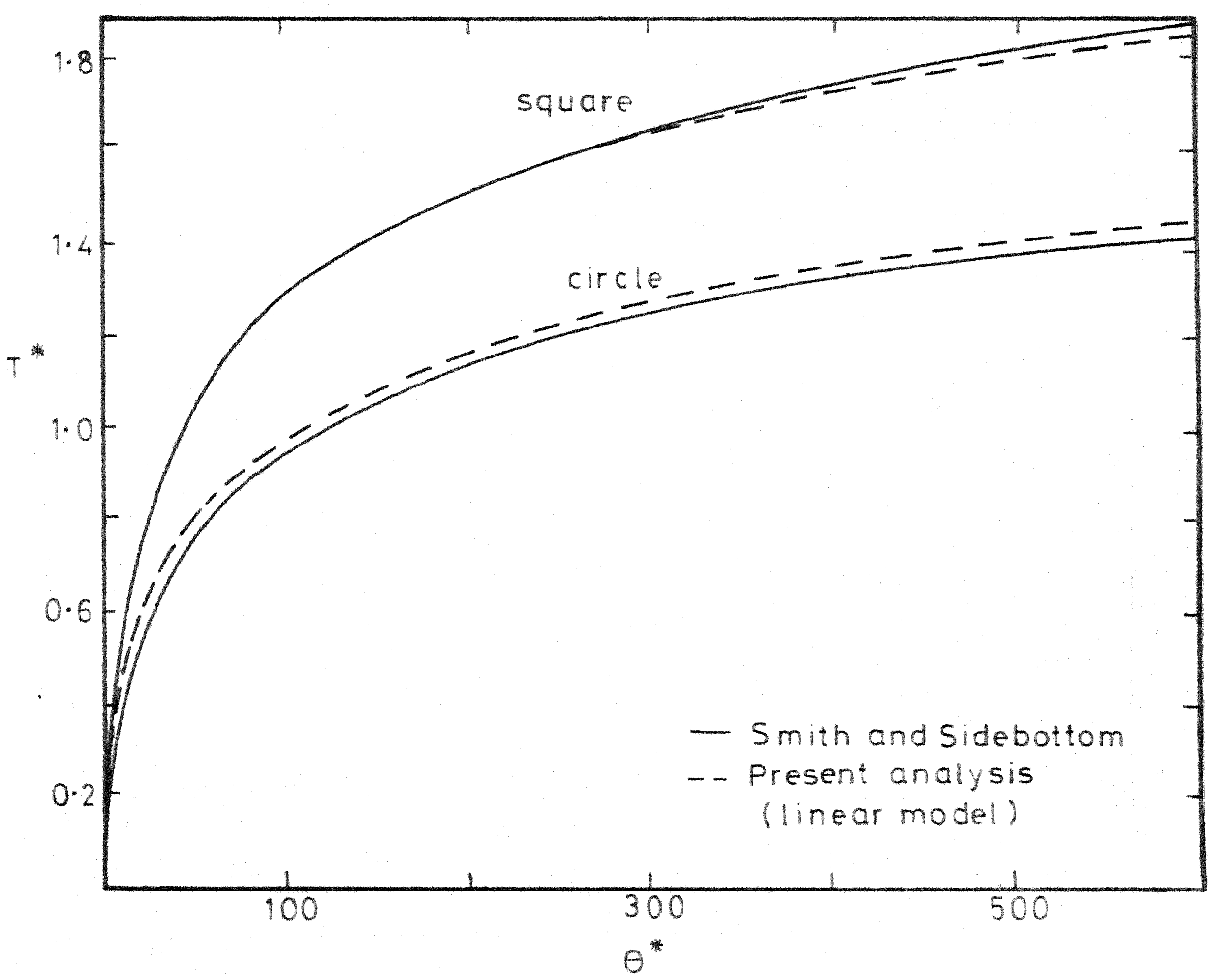


FIG. (5-9) T^* Vs θ^* FOR SQ. AND CIRCULAR SECTION

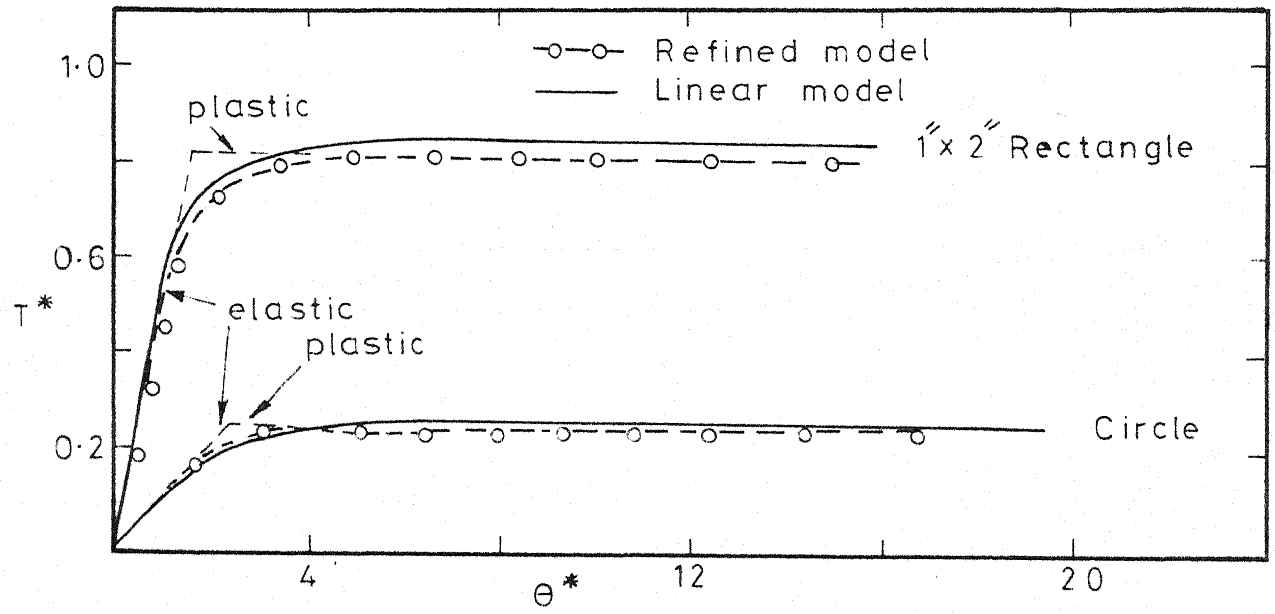


FIG (5-10) T^* Vs θ^* FOR RECTANGULAR AND CIRCULAR SECTION

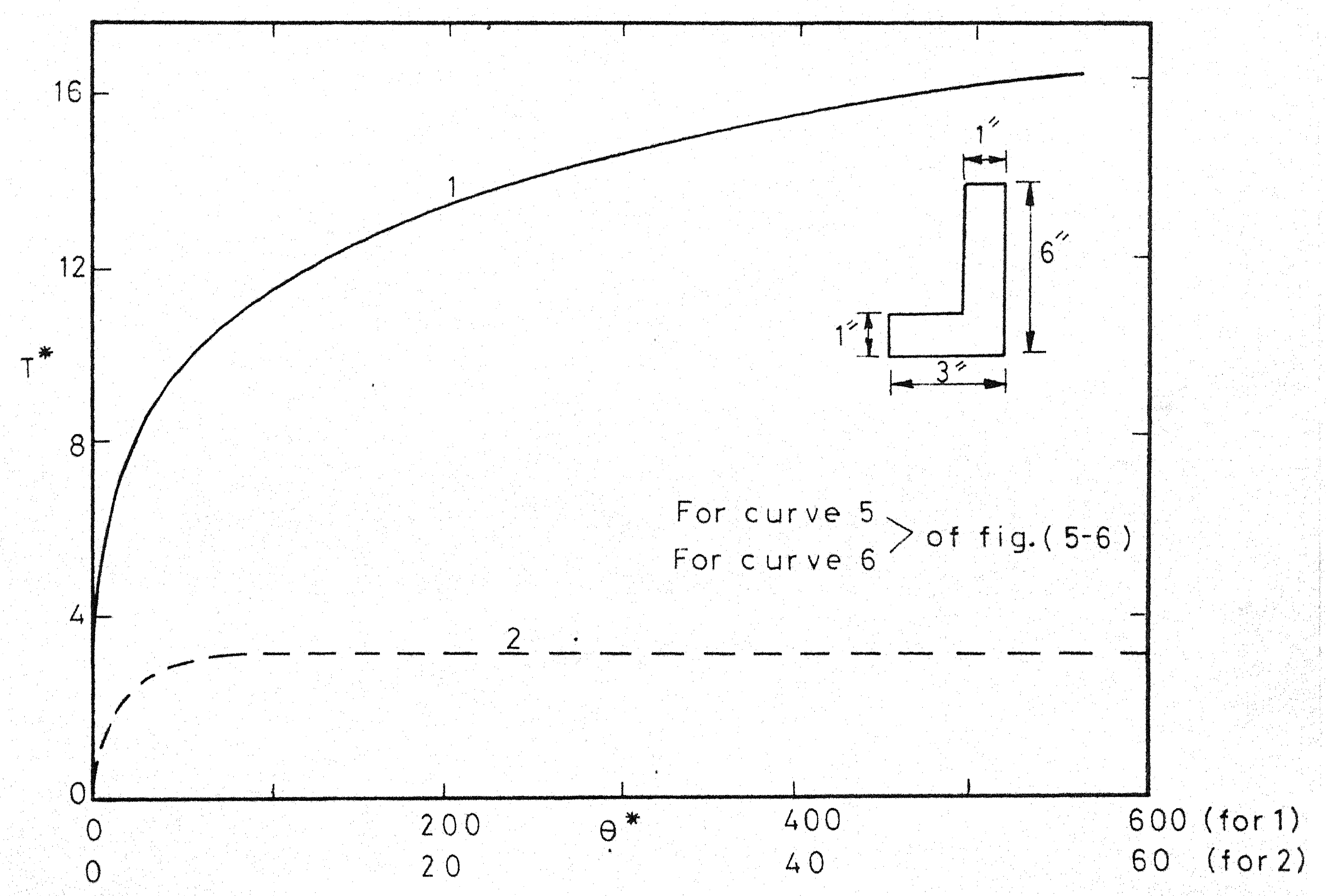
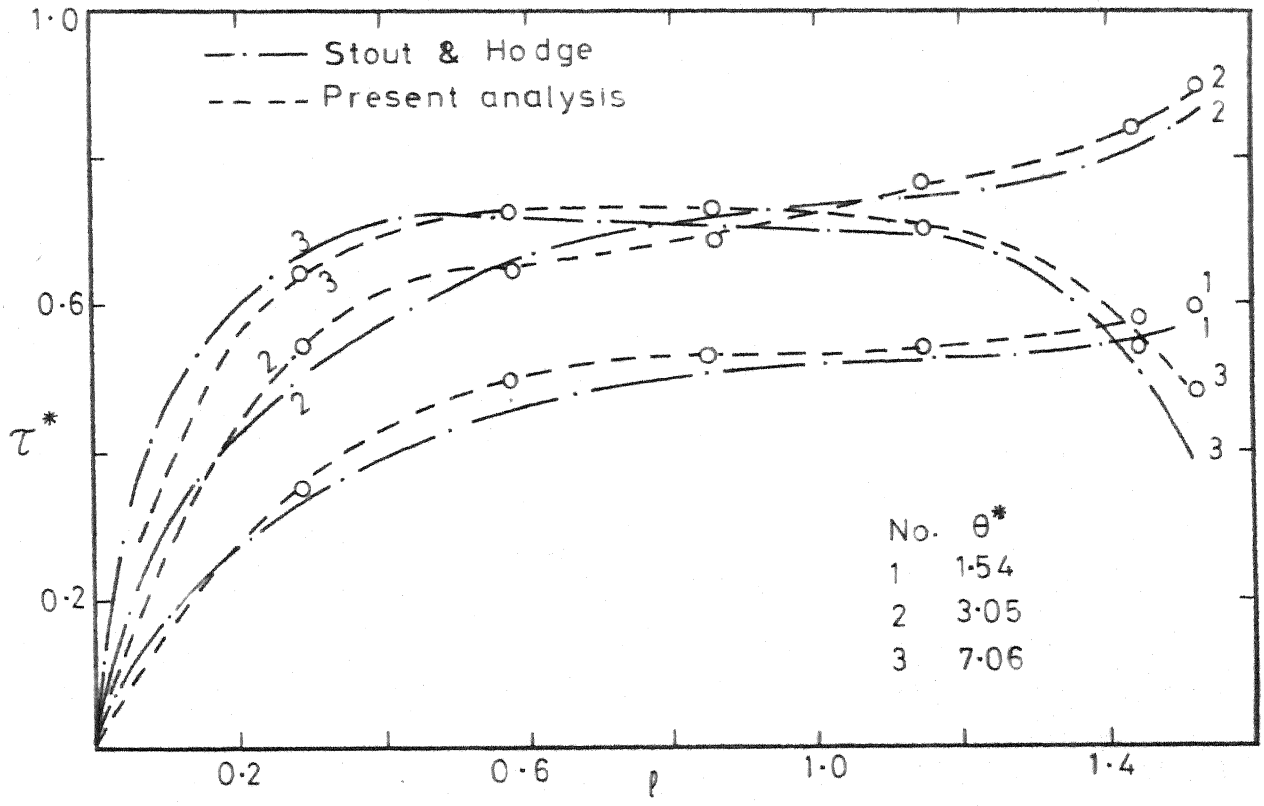
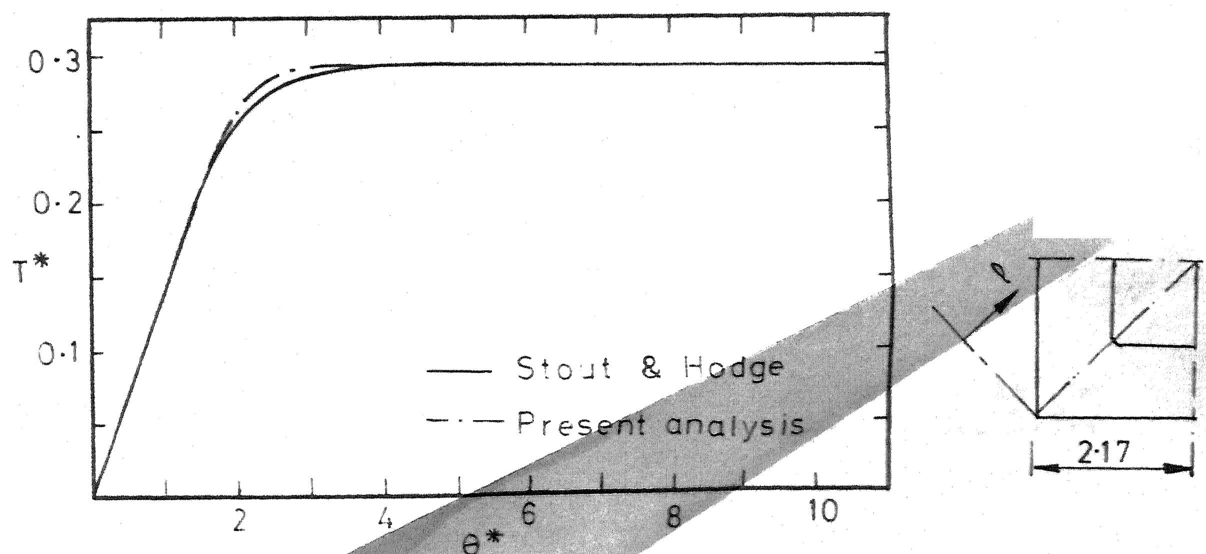


FIG. (5-11) T^* Vs θ^* FOR L-SECTION



FIG·(5-12b) RESULTANT STRESS DISTRIBUTION ALONG DIAGONAL



FIG·(5-12a) T^* Vs θ^* FOR HOLLOW SQ. CYLINDER

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CHAPTER 6

PLATE BENDING ANALYSIS

6.1 Introduction:

Structural plates of different shapes have a wide application in the aerospace industries. The finite element procedure is ideally suited for such general configurations since the plate may be approximated as a series of simple-shaped elements. Perhaps, the plate bending problem is the only field where the finite element method has been most widely applied through various element models. Earlier attempts for the analysis were confined to rectangular models and upto 1965, no satisfactory triangular model had been evolved. Failure of the previous investigations have been attributed due to lack of interelement compatibility and absence of completeness of the assumed distribution field which has been discussed earlier.

Notable contributors for rectangular displacement models are Adini and Clough[1], Bogner, Fox and Schmit[2], Dea'k and Pian[3] etc. Clough and Felippa[4] and Veubeke[5] have used quadrilateral elements which have very good convergence property. Successful triangular displacement models have been evolved by Clough and Tocher[6], Bazeley et. al.[7] and others; whereas the equilibrium models have been illustrated by Veubeke and Sanders[8]

and Morley[9]. Assumed stress hybrid model has been developed by Pian[10] and Severn and Taylor[11] applied this to rectangular as well as triangular models. Yamamoto[12] investigated the behavior of assumed displacement hybrid models using rectangular elements. The question of convergency for a displacement model has been discussed by Waltz et al.[13], whereas the necessary interelement continuity conditions and convergency for various model have been given by Pian and Tong[14].

For many engineering problems, a knowledge of the stress distribution is of as equal importance as the displacement field. Hence the finite element model should not only be capable of predicting displacements with reasonable accuracy, but also should be able to give a fairly accurate stress field. In some particular case, this may be achieved by choosing proper mesh sizes and adopting suitable averaging techniques. However, for many problems, this procedure may be impractical even with higher order displacement expansions. A likely procedure to over come this situation is to adopt hybrid models[10,12]. Even in this approach, due to discontinuities at the interfaces, the fidelity of the solution may be doubtful. Perhaps, mixed element model is best suited for this purpose.

Mixed element models have been studied by Herrmann[15,16] and Visser[17] for plate bending problems and by Prato[18] for general shells including the effects of shear deformation.

Herrmann[15] has developed a triangular model with linear distributions for transverse displacement and bending moments. In another paper[16], he has taken linear displacements and constant moments within each element and has continuity of normal moment at the interfaces. Even with this simple model he has shown that the method yields good results. Visser[17] has adopted a parabolic varying displacement field with linear moment distribution over a triangular element using Herrmann's[15] variational formulation. It has to be mentioned here that the shear forces in these approaches have to be calculated numerically and will be, at least, one order less accurate than the moment values even if interpolation is employed.

In this Chapter, a triangular mixed finite element model has been developed for plate bending problems in which shear effects are included. Linear distribution for all variables (transverse deflection, rotations, bending and twisting moments and transverse shears) is assumed and the resulting matrix equation is obtained by the use of Reissner's variational principle[19]. A detailed error analysis of the approximate equations is made and the convergence of the scheme is proved. This formulation has the advantage that (a) all quantities of interest are obtained directly; (b) error analysis is straight forward and hence corrections could be obtained; (c) extension of this formulation to shells of arbitrary shapes using curvilinear coordinates is feasible; (d) complications,

such as, gradual variations of thickness, temperature effects, initial stresses and dynamic formulation can be handled. The main disadvantage is that the total number of unknowns at a node point is higher than that of the existing formulations. This is partially compensated by the requirement of smaller number of triangles to achieve the same accuracy.

It may be noted here that though the plate equations have been derived through Reissner's energy approach, the governing equations which have been derived in Chapter 1 are equally applicable and when they will be simplified to this level, they will produce the same set of equations as those obtained through Reissner's energy. Since, the Reissner's approach is quite well known, it has been considered here.

6.2 Variational Formulation:

For linear theory of moderately thick plates, the variational formulation considering transverse shear deformation is fairly well known[19]. Hence without going into details, only the final form, relevant for subsequent derivations, would be presented. The notations used in the sequel have been illustrated in Fig. (6-1). The non-dimensionalised variational equation in condensed form is,

$$\delta J_R = \iint_A \{\delta \bar{D}^T (\bar{B}\bar{D} + \bar{L})\} \, dx \, dy - \int_{S_\sigma} \delta \hat{D}^T \hat{F} \, ds - \int_{S_d} \delta \tilde{F}^T \tilde{D} \, ds = 0 \tag{6.2-1}$$

where $\overline{B}$ is an 8 x 8 matrix of differential operators given by

$$\overline{B} = \begin{bmatrix} 0 & 0 & 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 & .1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & 0 & 0 & -12 & 0 & 12\nu & 0 & 0 \\ 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & 0 & 0 & -48(1+\nu) & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 12\nu & 0 & -12 & 0 & 0 \\ 1 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & -12(1+\nu)/5 & 0 \\ 0 & 1 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & -12(1+\nu)/5 \end{bmatrix} \quad (6.2\text{-}2)$$

the nondimensional state vector,

$$\overline{D}^T = \langle \beta_1 \; \beta_2 \; w \; M_1 \; M_{12} \; M_2 \; V_1 \; V_2 \rangle, \qquad (6.2\text{-}3)$$

the nondimensional load vector,

$$\overline{L}^T = \langle M_1 \; M_2 \; -q \; 0 \; 0 \; 0 \; 0 \; 0 \rangle \qquad (6.2\text{-}4)$$

The relation between actual (primed) and nondimensional quantities are given by,

$$\begin{aligned} x &= x'/h, \quad y = y'/h, \quad \beta_1 = \beta_1', \quad \beta_2 = \beta_2' \\ w &= w'/h \\ M_1 &= M_1'/Eh^2, \; M_2 = M_2'/Eh^2, \; M_{12} = M_{12}'/Eh^2 \qquad (6.2\text{-}5) \\ V_1 &= V'_1/Eh, \quad V_2 = V_2'/Eh, \\ m_1 &= m_1'/Eh, \; m_2 = m_2'/Eh \text{ and } q = q'/E \end{aligned}$$

where h is the thickness, E is Young's modulus and is

Poisson's ratio. It may be noted that the effect of normal stress is neglected here and hence the additional terms in $\bar{L}$ normally seen in the Reissner's formulation are omitted.

The surface integral in Eq. (6.2-1) extends over the entire plate, the first line integral is over the boundary, S_σ, where stresses are prescribed and the second line integral is over the boundary, S_d, where displacements are prescribed. The respective deformation and stress resultant vectors on S_σ and S_d are given by,

$$
\begin{aligned}
\hat{D}^T &= \langle \beta_n \; \beta_t \; w \rangle \quad \text{on } S_\sigma \\
\tilde{D}^T &= \langle w - w^* \;\; \beta_n - \beta_n^* \;\; \beta_t - \beta_t^* \rangle \text{ on } S_d \\
\hat{F}^T &= \langle M_n - M_n^* \;\; M_{nt} - M_{nt}^* \quad V_n - V_n^* \rangle \text{ on } S_\sigma \qquad (6.2\text{-}6) \\
\tilde{F}^T &= \langle V_n \; M_n \; M_t \rangle \quad \text{on } S_d
\end{aligned}
$$

where M_n, V_n and β_n are the nondimensional normal moment, shear and rotation along the edge respectively and the asterisked quantities are the corresponding prescribed non-dimensional values. The Euler Equations of (6.2-1) will furnish the first three equations as differential equations of equilibrium and the next five equations as force deformation relations. The resulting eight equations can be written in matrix form as,

$$\bar{B}\,\bar{D} + \bar{L} = 0 \qquad (6.2\text{-}7)$$

The independent vanishing of the line integrals will provide the required boundary conditions.

$$M_n = M_n^*, \quad M_{nt} = M_{nt}^* \text{ and } V_n = V_n^* \text{ on } S_\sigma \qquad (6.2\text{-}8)$$

and $$w = w^*, \quad \beta_n = \beta_n^* \text{ and } \beta_t = \beta_t^* \text{ on } S_d$$

6.3 Finite Element Formulation:

Within each element, linear distribution has been adopted for the approximate state variable vector, D. It may be noted that the total number of variables used here is larger than that required for a compatible displacement model. However, it has the advantage that, because of linear distribution, continuity of deflection, bending slopes and stresses will be obtained and due to complete polynomial distribution function, completeness and invariance criteria[14,20] will be automatically satisfied. However, there will be a limitation for this model, when the actual stress field is discontinuous. An indirect approach has been discussed to overcome this difficulty.

To facilitate the choice of trial function, the plate is represented by R number of triangular elements having a total number of nodes equal to P. A typical node, denoted by p, will be common to M number of elements, as shown in Fig. (6-2). For an element, the nodes are designated as 1,2 and 3. Local coordinates and dimensions of the element have been shown in Fig. (6-3). A typical quantity, H, at any

point (x^1, x^2) inside an element, m, may be expressed in terms of its nodal values as,

$$H = \phi^T \, t \, H^m \tag{6.3-1}$$

where

$$\phi^T = \langle 1 \;\; x^1/a \;\; x^2/a \rangle \tag{6.3-2}$$

$$H^m = \langle H_1 \;\; H_2 \;\; H_3 \rangle^m \tag{6.3-3}$$

$$t = \frac{1}{2A} \begin{bmatrix} (2/3)A & (2/3)A & (2/3)A \\ (a_{22}-a_{32})/a & a_{32}/a & -a_{22}/a \\ (a_{31}-a_{21})/a & -a_{31}/a & a_{21}/a \end{bmatrix} \tag{6.3-4}$$

$$2A = (a_{21}\, a_{32} - a_{31}\, a_{22})/a^2,$$

and 'a' is a typical dimension of the plate. Following Eq. (6.3-1) the distribution of the state vector in terms of its nodal value may be given as,

$$D = \Phi \, T \, \Omega \, D^m \tag{6.3-5}$$

where

$$D^T = \langle \beta_1 \;\; \beta_2 \;\; w \;\; M_1 \;\; M_{12} \;\; M_2 \;\; V_1 \;\; V_2 \rangle \tag{6.3-6}$$

$$\Phi = \text{Dia} \lceil \phi^T \;\; \phi^T \;\; \phi^T \;\; \phi^T \;\; \phi^T \;\; \phi^T \;\; \phi^T \;\; \phi^T \rfloor \tag{6.3-7}$$

$$T = \text{Dia} \lceil t \;\; t \;\; t \;\; t \;\; t \;\; t \;\; t \;\; t \rfloor \tag{6.3-8}$$

$$D^{m^T} = \langle D_1^T \;\; D_2^T \;\; D_3^T \rangle \tag{6.3-9}$$

where D_i is D at the node, i. Matrix Ω is a 24 x 24. boolean transformation matrix to rearrange the elements of the vector.

$$\langle \beta_{11} \;\; \beta_{12} \;\; \beta_{13} \;\; \beta_{21} \;\; \beta_{22} \;\; \beta_{23} \;\; \cdots \;\; V_{13} \;\; V_{21} \;\; V_{22} \;\; V_{23} \rangle \text{ to } D^{m^T}$$

In the above, the first subscript denotes coordinate direction and the second one corresponds to the node number of the element. It should be noted that $\overline{D}$ and D are not identical. Vector $\overline{D}$ represents the exact solution surfaces and is the solution of the differential equation (6.2-7) with boundary conditions (6.2-8); whereas, D is the approximate solution of the finite element method and is valid only within the m-th element. Hence the exact state vector $\overline{D}$ for element m, may be written as,

$$\overline{D} = \Phi\, T\, \Omega\, D^m + \epsilon^m \tag{6.3-10}$$

where, ϵ^m is the associated error vector for the element m. Substituting $\overline{D}$ from Eq. (6.3-10) in the surface integral portion of Eq. (6.2-1), we get for the element m, the expression,

$$\delta\, D^{m^T} (|K^m D^m + L^m| + E^m) + \iint_{A_m} \delta\epsilon^{m^T}\, |\,\overline{\overline{B}}\,\overline{D} + \overline{L}\,|\, dx^1\, dx^2 \tag{6.3-11}$$

where $$K^m = \Omega^T T^T \left(\iint_{A_m} \Phi^T\, \overline{B}\; dx^1\, dx^2\right) T\, \Omega \tag{6.3-12}$$

$$L^m = \Omega^T T^T \left(\iint_{A_m} \Phi^T\, \overline{L}\; dx^1\, dx^2\right) \tag{6.3-13}$$

$$E^m = \Omega^T T^T \left(\iint_{A_m} \Phi^T\, \overline{B}\, \epsilon^m\, dx^1\, dx^2\right) \tag{6.3-14}$$

and A_m is the area of the m-th element. The 24 x 24 matrix, K^m, and, for linear distribution of load, the exact expression for L^m may be calculated numerically. The expression

(6.3-11) can be written in a slightly different form as,

$$\sum_{i=1,2,3} \langle \delta D_i^{m^T} \rangle \left| \sum_{j=1,2,3} K_{ij}^m D_j^m + L_i^m + E_i^m \right|$$

$$+ \iint_{A_m} \delta \epsilon^{m^T} \left| \overline{B}\,\overline{D} + \overline{L} \right| dx^1\, dx^2 \qquad (6.3\text{-}15)$$

where $L^{m^T} = \langle L_1^{m^T}\ L_2^{m^T}\ L_3^{m^T} \rangle$

$$E^{m^T} = \langle E_1^{m^T}\ E_2^{m^T}\ E_3^{m^T} \rangle \qquad (6.3\text{-}16)$$

and partitioning of K^m and L^m matrices have been given in Tables (6-1) to (6-5).

Performing summation over all the elements after appropriate superposition and noting that there are only M elements surrounding a typical p-th node, the expression (6.3-15) gives,

$$\sum_{p=1,\dots P} \delta D_p^T \left| \sum_{m=1}^{M} \left(\sum_{j=1}^{3} K_{ij}^m D_J^m + L_i^m + E_i^m \right) \right|$$

$$+ \sum_{m=1,..R} \iint_{A_m} \delta \epsilon^{m^T} \left| \overline{B}\,\overline{D} + \overline{L} \right| dx^1\, dx^2$$

$$(6.3\text{-}17)$$

where the i-th node of the element m is p.

The independent vanishing of the coefficient of δD_p^T in (6.3-17) gives,

$$\sum_{m=1,\dots M} \left(\sum_{j=1,2,3} K_{ij}^m D_j^m + L_i^m + E_i^m \right) = 0 \qquad (6.3\text{-}18)$$

and vanishing of $\delta \epsilon_m^T$ gives Eq. (6.2-7).

Consider the line integral in Eq. (6.2-1). Designate a typical quantity by H the conjugate quantity (with respect to work) by γ and associated error terms by ϵ^H and ϵ^γ. Integrating from node p to p + 1 along the boundary whose outer normal subtends an angle θ with x^1 axis, as shown in Fig. (6-1), a typical line integral may be written as,

$$\int_s \left| H^* - H - \epsilon^H \right| \delta\,(\gamma + \epsilon^\gamma)\, ds \qquad (6.3\text{-}19)$$

where H^* is the specified value of H along s. Since H and γ are assumed to be linear functions, the expression (6.3-19) may be written as,

$$\left| \int_s (H^* - H_p - \frac{H_{p+1} - H_p}{s_p} s - \epsilon^H)(1 - \frac{s}{s_p})\, ds \right| \delta\gamma_p$$

$$+ \left| \int_s (H^* - H_p - \frac{H_{p+1} - H_p}{s_p} s - \epsilon^H) \frac{s}{s_p}\, ds \right| \delta\gamma_{p+1}$$

$$+ \int_s \delta\,\epsilon^\gamma \;(H^* - H)\, ds \qquad (6.3\text{-}20)$$

where s_p is the length of the path of integration from p to p + 1 and the barred quantity is the corresponding exact value on boundary s. Since the distribution of H^* is known, integrations for the coefficient of $\delta\gamma_p$ and $\delta\gamma_{p+1}$ may be easily performed. It is to be noted that independent vanishing of the coefficient for $\delta\epsilon^\gamma$ will give the exact boundary condition (6.2-8). Using standard transformation relations to convert the normal and tangential components (n - t coordinate system) to usual quantities, in (x^1, x^2 system) the

coefficient of $\delta\gamma_p$ may be appropriately added to Eq. (6.3-18) to impose necessary boundary constraints.

Neglecting the error terms, the reduced matrix equation for the entire assemblage of elements which is of a banded nature, may be written as,

$$\underset{\sim}{K}\ \underset{\sim}{D} + \underset{\sim}{L} = 0 \tag{6.3-21}$$

where, $\underset{\sim}{K}$ is the overall constrained 'stiffness' matrix,

$$\underset{\sim}{D}^T = \langle D_1^T \ D_2^T \ \dots \ D_p^T \rangle,$$

$$\underset{\sim}{L}^T = \langle L_1^T \ L_2^T \ \dots \ L_P^T \rangle \tag{6.3-22}$$

Solving Eq. (6.3-21) by standard algorithms, such as the Gauss elimination procedure, the approximate vector $\underset{\sim}{D}$ may be obtained.

6.4 Error Analysis:

Accuracy and convergence study of a proposed finite element model is generally based on numerical evaluation of some problems and comparing the solution with the available analytical results. Such a procedure may be valuable in providing a qualitative character of different models, but is basically deficient in general mathematical justification. J.E. Walz et. al.[13] have investigated convergence properties of several finite element displacement models based on classical order of error analysis.

In the following, convergence and error properties of the proposed model are studied to obtain the approximate discretization error involved in the analysis. The discretization error is due to the omission of the term E_i^m in Eq.(6.3-18), which in turn, is a consequence of the approximation for the distribution of the state variables.

By Taylor series expansion, the functional value of $\overline{D}$ at any point α (α = 1, 2, 3) can be expressed in terms of $\overline{D}$ and its derivatives at point i(i = 1, 2, and 3). Subtracting appropriate error terms at the nodes, the approximate state vector D can be written as,

$$D_\alpha = D_i + \epsilon_i - \epsilon_\alpha + \sum_{j=1,2} x_j^{\alpha i} \left(\frac{\partial \overline{D}}{\partial x^j} \right)_i + \sum\sum_{j,k=1,2} x_j^{\alpha i} x_k^{\alpha i} \left(\frac{\partial^2 \overline{D}}{\partial x^j \, \partial x^k} \right)_i + \ldots \tag{6.4-1}$$

where $x_j^{\alpha i} = a_{\alpha j} - a_{ij}$ and ϵ_i is discretization error at the node i. Concentrating the attention on node p (recalling p is the i-th node of element m), substituting (6.4-1) in Eq. (6.3-18) and since the equation,

$$\sum_{m=1,..M} \sum_{j=1,2,3} (K_{ij}^m D_j^m + L_i^m) = 0$$

has been already solved for approximate solution, it can be shown that,

$$\sum_{m=1,..M} \sum_{j=1,2,3} K_{ij}^m (\epsilon_i - \epsilon_j) + E_i^m + \sum_{\alpha=1,2,3} \sum_{j=1,2} x_j^{\alpha i} K_{i\alpha} \left(\frac{\partial \epsilon}{\partial x^j} \right)_i + \sum_{j=2,3..\infty} A_j = 0 \tag{6.4-2}$$

where A_j's are given by,

$$A_2 = \frac{1}{2!} \sum_{\alpha=1,2,3} \sum\sum_{j,k=1,2} x_j^{\alpha i}\, x_k^{\alpha i}\, K_{i\alpha}^m \left(\frac{\partial^2 \bar{D}}{\partial x^j\, \partial x^k} \right)_i$$

$$A_3 = \frac{1}{3!} \sum_{\alpha=1,2,3} \sum\sum\sum_{j,k,l=1,2} x_j^{\alpha i}\, x_k^{\alpha i}\, x_l^{\alpha i}\, K_{i\alpha}^m \left(\frac{\partial^3 \bar{D}}{\partial x^j \partial x^k \partial x^l} \right)_i \tag{6.4-3}$$

and other A's can be similarly written. For approximate error analysis, the vector ϵ^m may be approximated in a linear form similar to that of the state variable vector D. Following this approximation, Eqs. (6.3-14) and (6.3-16) will give,

$$E_i^m = K_{i1}^m\, \epsilon_1^m + K_{i2}^m + K_{i3}^m\, \epsilon_3^m \tag{6.4-4}$$

Substituting (6.4-4) in Eq. (6.4-2) and differentiating the term $(\partial\epsilon/\partial x^j)_i$ for linear distribution, it may be shown that,

$$\sum_{m=1,\ldots M} \left| K_{i1}^m\, \epsilon_1^m + K_{i2}^m\, \epsilon_2^m + K_{i3}^m\, \epsilon_3^m + \sum_{j=2,\ldots\alpha} A_j \right| = 0 \tag{6.4-5}$$

Eq. (6.4-5) can be designated as the unconstrained equation for the error vector at the node p, and the expression in the braces as pseudo load. These can be calculated approximately from the known value of D by any numerical technique. Consider now the part of the expression (6.3-19) involving $\delta\gamma_p$. Following the procedure adopted for the surface integral, the expression containing $\delta\gamma_p$ in Eq. (6.3-19) can be written as,

$$\delta\gamma_p \left| \int_s \epsilon^H \left(1 - \frac{s}{s_p}\right) ds + \frac{s_p}{6} (\epsilon^H{}_p - \epsilon^H{}_{p+1} \right.$$

$$\left. + \left(\frac{\partial\bar{\zeta}}{\partial s}\right)_p s_p + \frac{1}{2!} \left(\frac{\partial^2\bar{\zeta}}{ds^2} \right) s_p^2 + \ldots \right| \qquad (6.4\text{-}6)$$

where $\bar{\zeta}$ is the exact expression for H^*-H on s. Again, for approximate solution of error, taking linear distribution we get,

$$\delta\,\gamma_p \frac{s_p}{6} \left| 3\epsilon_p^H + \left(\frac{\partial\bar{\zeta}}{ds}\right)_p s_p + \frac{1}{2!} \left(\frac{d^2\bar{\zeta}}{ds^2}\right) s_p^2 + \ldots \right| \quad (6.4\text{-}7)$$

Since the derivatives of $\bar{\zeta}$ can be calculated approximately from the previous solution, this expression (after applying proper transformation to (x^1, x^2) coordinate system) has to be added for appropriate boundary to the Eq. (6.4-5) as constraint. Hence the resulting reduced error equation can be solved.

To show the characteristics of the model more clearly all a_{ij}'s Eq. (6.4-5) may be divided by a scale factor λ where,

$$a_{ij} = \lambda\, a'_{ij}$$

Such that when λ tends to zero, all the M number of nodes surrounding a typical node p, tend to shrink to p, keeping the relative dimensions of each triangular element (a'_{ij}) to be the same. It can be easily shown that when $\lambda \to 0$, Eq. (6.4-5) is of the form,

$$\lim_{\lambda \to 0} \iint_{A} \left| (\bar{E}\,\epsilon) + \lambda\{\ \} + \lambda^2\{\ \} + \ldots \right| dA = 0 \tag{6.4-8}$$

It is seen from (6.4-3) that expressions in the braces always remain bounded as λ tends to zero and hence Eq. (6.4-8) will yield trivial solution $\epsilon = 0$. (Actually, it will be only solution from the uniqueness consideration of elasticity). Convergence to the true solution is thus assured. Monotonicity will be achieved if the lower order derivatives in the pseudo loading terms retain same sign in the region bounded by the M elements as the element size (λ) is reduced.

It can be shown that for regular element configuration, the coefficient multiplied by λ in the load term is identically zero. Thus, the solution converges quadratically when elements are regular. Also, if the state variable at the nodes of an element are expressed in terms of variables at the centroid of the element by Taylor's series expansion, then, it is seen that the coefficient of λ again vanishes, thereby indicating that the solution at the centroid of the element is better behaved and converges quadratically. This result has been demonstrated by several authors.

Finally, near the boundary region, the solution converges linearly because of the presence of the linear term in λ .

Similar conclusions can be drawn for the expression (6.4-7) which shows the error committed in discretizing the state vector at the boundary of the plate.

6.5 Discontinuity of Stress Field:

Discontinuity of stress field in the elastic range may result from the following situations.

(a) Where there is an abrupt change in the thickness of the plate.

(b) When the plate is made up of different materials.

Across the discontinuities, the normal moment and shear are continuous, whereas the tangential components of the stresses are discontinuous. Independent tangential variables may be assumed on either side of the discontinuity and the corresponding equations may be obtained by the independent vanishing of the expressions multiplying the variations of these tangential values. The result is that the number of unknowns and the corresponding number of equations increase at such discontinuities. Also, for such cases, the band width of the resulting simultaneous equations Eq. (6.3-21) would increase.

6.6 Numerical Results and Conclusions:

6.6.1 Numerical Results:

To show numerical convergence and error properties several problems have been solved.

Fig. (6-5) shows deflection w', moment M_1', and shear, V_2', for a simply supported square plate having 2, 3, 4 and 6 divisions in a quarter of a plate as shown in Fig. (6-4) (Example 1). It is seen that, inspite of the large

element sizes, the results are well behaved and close to the correct solution. Monotonicity of results is clearly seen.

Fig. (6-6) shows deflection, w', and moment, M_1', for a rectangular plate with opposite edges simply supported, one edge fixed and the other edge free (Example 2). Four sub-divisions have been taken for one-half of the plate. This example has been taken from Herrmann[15,16]. The solutions are compared with the series solution given in the references 15 and 16. Except near the fixed edges, the finite element solution shows excellent agreement with the series solution.

Fig. (6-7) shows deflection w', moments M_1', M_2', and shear V_2' for a square plate with two opposite edges clamped and the other two simply supported for thickness to span ratio 0.01 and 0.1. Except the moments near the fixed edge, other quantities are very well behaved and quite close to the actual solution. Exact solutions for deflections, moments and shears have been drawn for the thin plate[21]. In the finite element solutions, differences of moments and shears between the thick and thin plates have been obtained between 2 to 3 percent except, only, in the case of shear V_2' along $y = a/2$, where the maximum deviation is about 11 percent. Exact value of the maximum deflection and moment for a thick simply supported plate has been taken from the reference 22.

Lastly, error corrections have been applied using Eq. (6.4-3) to the finite element solutions and the results

have been shown in Table (6-5). Since the error at any node has been assumed to be the maximum of the corresponding error anywhere in the plate, the wide deviation of the corrected value seems to be justified. Nevertheless, the corrections furnish a bound which gives an approximate idea for the location of the exact solution.

6.6.2 Conclusions:

A mixed finite element model for plate problems has been developed. This model takes into account the effects due to shear deformation. Because of the linear distribution of the state variables, inter element compatibility is completely satisfied. Equilibrium and compatibility conditions within an element are satisfied 'in the mean' through the Reissner variational principle. Since the state variables contain all quantities of interest to the designer, no further effort or approximation is needed to evaluate these.

Error analysis based on the classical order of accuracy approach is developed. Convergence to the true solution has been proved and is shown by numerical experiments. The order of accuracy developed here is useful to obtain a 'deferred approach to the limit'. A few numerical results have been presented to show the effectiveness of the proposed model. Solutions have also been obtained for various shapes and boundary conditions of the plate and the results agree closely with those published in the literature.

$$K^m = \begin{bmatrix} K^m_{11} & K^m_{12} & K^m_{13} \\ K^m_{21} & K^m_{22} & K^m_{23} \\ K^m_{31} & K^m_{32} & K^m_{33} \end{bmatrix}$$

Notations:

$$b_{32} = (a_{31} - a_{21})/\ 6a^2$$
$$b_{13} = -a_{31}/6a^2$$
$$b_{21} = a_{21}/6a^2$$
$$S_1 = 12$$
$$c_{23} = (a_{22}-a_{32})/6a^2$$
$$c_{31} = a_{32}/6a^2$$
$$c_{12} = -a_{22}/6a^2$$
$$S_2 = 48\ (1 + \nu)$$
$$d_1 = A/6$$
$$d_2 = A/12$$
$$S_3 = 12\ (1 + \nu)/5$$

Table 6-1 · Partition of Matrix K^m and notations.

$$K_{11}^{m}=a^{2}\begin{bmatrix} & & & -C_{23} & -b_{32} & & d_1 & \\ & & & & -C_{23} & -b_{32} & & d_1 \\ & & & & & & -C_{23} & -b_{32} \\ C_{23} & & & -S_1d_1 & & \nu S_1d_1 & & \\ 2b_{32} & 2C_{23} & & & -S_2d_1 & & & \\ & b_{32} & & \nu S_1d_1 & & -S_1d_1 & & \\ d_1 & & C_{23} & & & & -S_2d_1 & \\ & d_1 & b_{32} & & & & & -S_3d_1 \end{bmatrix}$$

$$K_{12}^{m}=a^{2}\begin{bmatrix} & & & -C_{31} & -b_{13} & & d_2 & \\ & & & & -C_{31} & -b_{13} & & d_2 \\ & & & & & & -C_{31} & -b_{13} \\ C_{31} & & & -S_1d_2 & & \nu S_1d_2 & & \\ 2b_{13} & 2C_{31} & & & -S_2d_2 & & & \\ & b_{13} & & \nu S_1d_2 & & -S_1d_2 & & \\ d_2 & & C_{31} & & & & -S_3d_2 & \\ & d_2 & b_{13} & & & & & -S_3d_2 \end{bmatrix}$$

$$K_{13}^{m}=a^{2}\begin{bmatrix} & & & -C_{12} & -b_{21} & & d_2 & \\ & & & & -C_{12} & -b_{21} & & d_2 \\ & & & & & & -C_{12} & -b_{21} \\ C_{12} & & & -S_1d_2 & & \nu S_1d_2 & & \\ 2b_{21} & 2C_{12} & & & -S_2d_2 & & & \\ & b_{21} & & \nu S_1d_2 & & -S_1d_2 & & \\ d_2 & & C_{12} & & & & -S_3d_2 & \\ & d_2 & b_{21} & & & & & -S_3d_2 \end{bmatrix}$$

Table 6-2 : Matrices K_{11}^{m}, K_{12}^{m} and K_{13}^{m}

$$K^m_{21}=a^2\begin{bmatrix} & & & -C_{23} & -b_{32} & & d_2 & \\ & & & & -C_{23} & -b_{32} & & d_2 \\ & & & & & & C_{23} & b_{32} \\ C_{23} & & & -S_1d_2 & & \nu S_1d_2 & & \\ 2b_{32} & 2C_{23} & & & -S_2d_2 & & & \\ & b_{32} & & \nu S_1d_2 & & -S_1d_2 & & \\ d_2 & & C_{23} & & & & -S_3d_2 & \\ & d_2 & b_{32} & & & & & -S_3d_2 \end{bmatrix}$$

$$K^m_{22}=a^2\begin{bmatrix} & & & -C_{31} & -b_{13} & & d_1 & \\ & & & & -C_{31} & -b_{13} & & d_1 \\ & & & & & & -C_{31} & -b_{13} \\ C_{31} & & & -S_1d_1 & & \nu S_1d_1 & & \\ 2b_{13} & 2C_{31} & & & -S_2d_1 & & & \\ & b_{13} & & \nu S_1d_1 & & -S_1d_1 & & \\ d_1 & & C_{31} & & & & -S_3d_1 & \\ & d_1 & b_{13} & & & & & -S_3d_1 \end{bmatrix}$$

$$K^m_{23}=a^2\begin{bmatrix} & & & -C_{12} & -b_{21} & & d_2 & \\ & & & & -C_{12} & -b_{21} & & d_2 \\ & & & & & & -C_{12} & -b_{21} \\ C_{12} & & & -S_1d_2 & & \nu S_1d_2 & & \\ 2b_{21} & 2C_{12} & & & -S_2d_2 & & & \\ & b_{21} & & \nu S_1d_2 & & -S_1d_2 & & \\ d_2 & & C_{12} & & & & -S_3d_2 & \\ & d_2 & b_{21} & & & & & -S_3d_2 \end{bmatrix}$$

Table 6-3: Matrices K^m_{21}, K^m_{22}, K^m_{23}

$$K^m_{31}=a^2\begin{bmatrix} & & & -C_{23} & -b_{32} & & d_2 & \\ & & & & -C_{23} & -b_{32} & & d_2 \\ & & & & & & C_{23} & b_{32} \\ C_{23} & & & -S_1d_2 & & \nu S_1d_2 & & \\ 2b_{32} & 2C_{23} & & & -S_2d_2 & & & \\ & 2b_{32} & & \nu S_1d_2 & & -S_1d_2 & & \\ d_2 & & C_{23} & & & & -S_3d_2 & \\ & d_2 & b_{32} & & & & & -S_3d_2 \end{bmatrix}$$

$$K^m_{32}=a^2\begin{bmatrix} & & & -C_{31} & -b_{13} & & d_2 & \\ & & & & -C_{31} & -b_{13} & & d_2 \\ & & & & & & -C_{31} & -b_{13} \\ C_{31} & & & -S_1d_2 & & \nu S_1d_2 & & \\ 2b_{13} & 2C_{31} & & & -S_2d_2 & & & \\ & b_{13} & & \nu S_1d_2 & & -S_1d_2 & & \\ d_2 & & C_{31} & & & & -S_3d_2 & \\ & d_2 & b_{13} & & & & & -S_3d_2 \end{bmatrix}$$

$$K^m_{33}=a^2\begin{bmatrix} & & & -C_{12} & -b_{21} & & d_1 & \\ & & & & -C_{12} & -b_{21} & & d_1 \\ & & & & & & -C_{12} & -b_{21} \\ C_{12} & & & -S_1d_1 & & \nu S_1d_1 & & \\ 2b_{21} & 2C_{12} & & & -S_2d_1 & & & \\ & b_{21} & & \nu S_1d_1 & & -S_1d_1 & & \\ d_1 & & C_{12} & & & & -S_3d_1 & \\ & d_1 & d_{21} & & & & & -S_3d_1 \end{bmatrix}$$

Table 6-4: Matrices K^m_{31}, K^m_{32} and K^m_{33}.

$$
L^m = \begin{Bmatrix} L_1^m \\ L_2^m \\ L_3^m \end{Bmatrix} = a^2 \begin{Bmatrix}
-d_1 m_{11} & -d_2 m_{12} & -d_2 m_{13} \\
-d_1 m_{21} & -d_2 m_{22} & -d_2 m_{23} \\
d_1 q_1 & d_2 q_2 & d_2 q_3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-d_2 m_{11} & -d_1 m_{12} & -d_2 m_{13} \\
-d_2 m_{21} & -d_1 m_{22} & -d_2 m_{23} \\
d_2 q_1 & d_1 q_2 & d_1 q_3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-d_2 m_{11} & -d_2 m_{12} & -d_1 m_{13} \\
-d_2 m_{21} & -d_2 m_{22} & -d_1 m_{23} \\
d_2 q_1 & d_2 q_2 & d_1 q_3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{Bmatrix}
$$

Table (6-5): Load Vector.

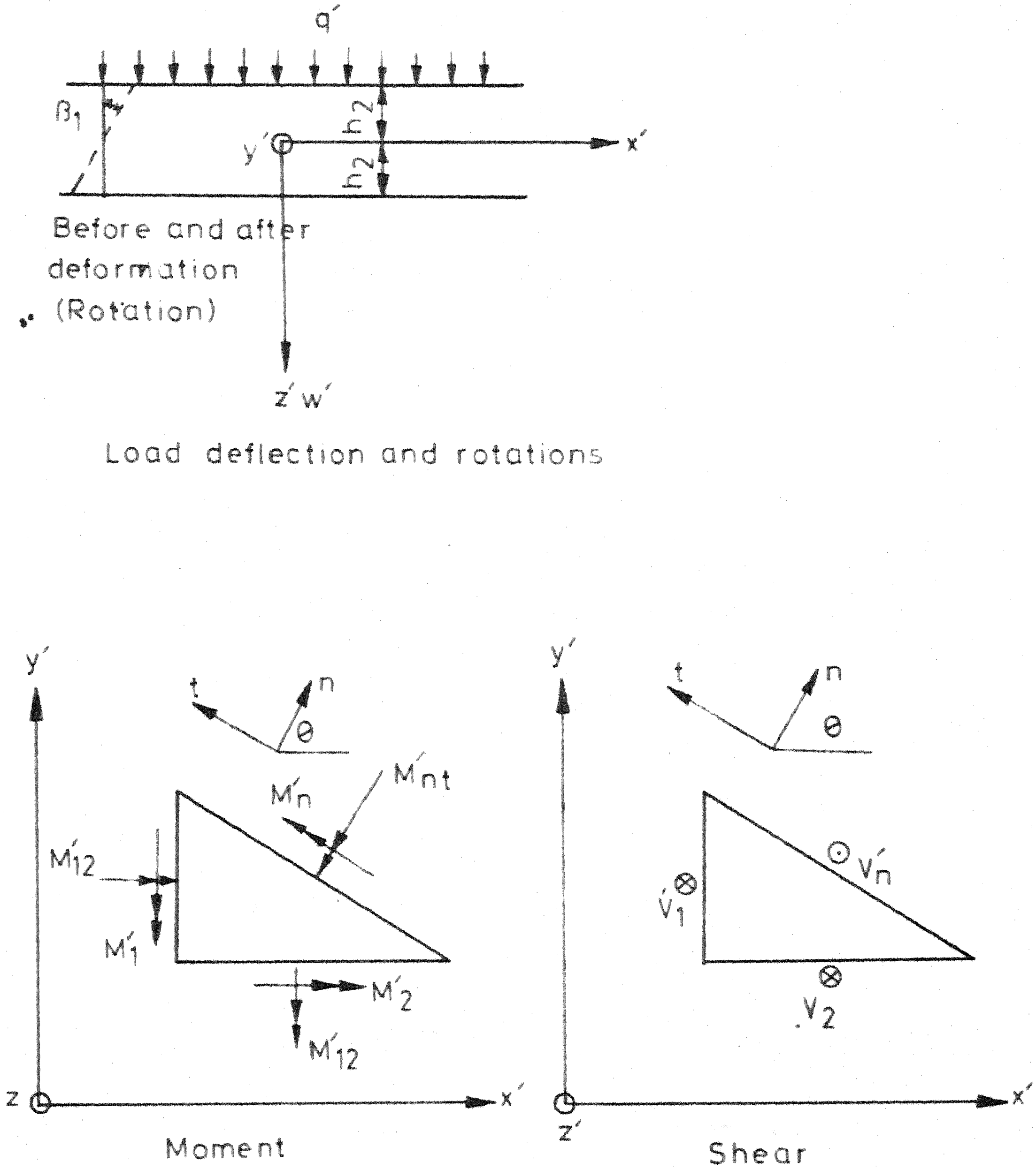


FIG. (6-1) NOTATIONS

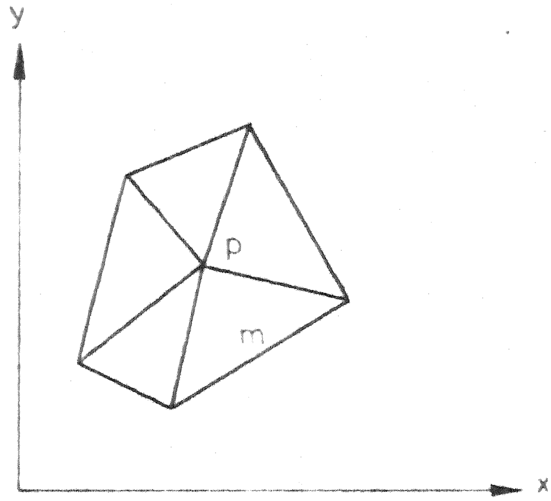


FIG. (6-2) TYPICAL ELEMENT m, TYPICAL NODE P COMMON TO M ELEMENTS

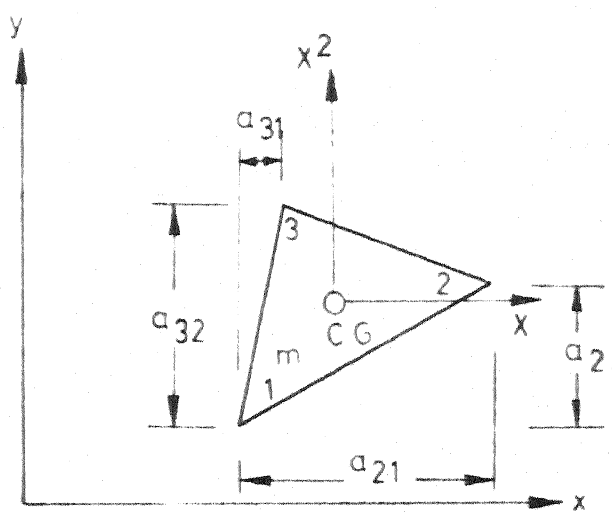
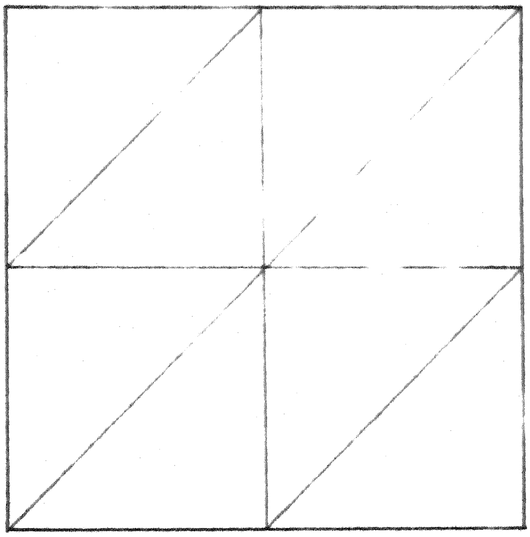
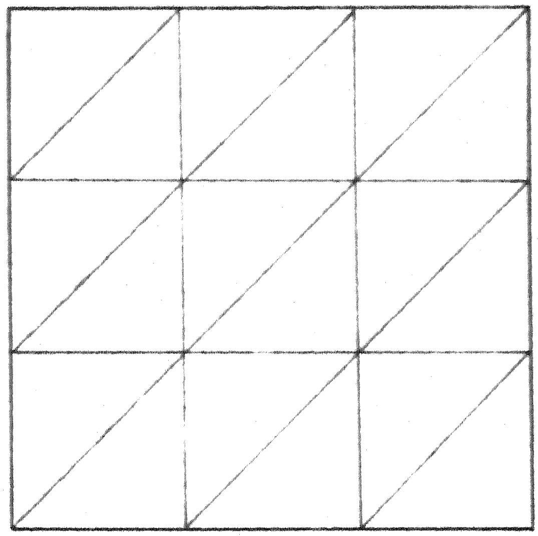


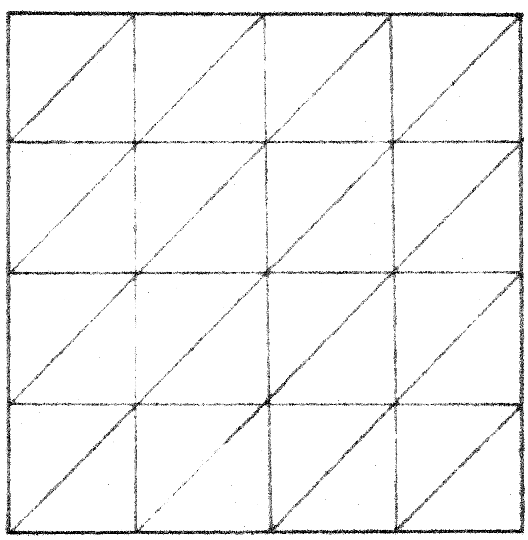
FIG. (6-3) LOCAL COORDINATES AND - DIMENSIONS OF AN ELEMENT



(a) 2 Divisions



(b) 3 Divisions



(c) 4 Divisions

FIG. (6-4) ARRANGEMENTS OF ELEMENTS

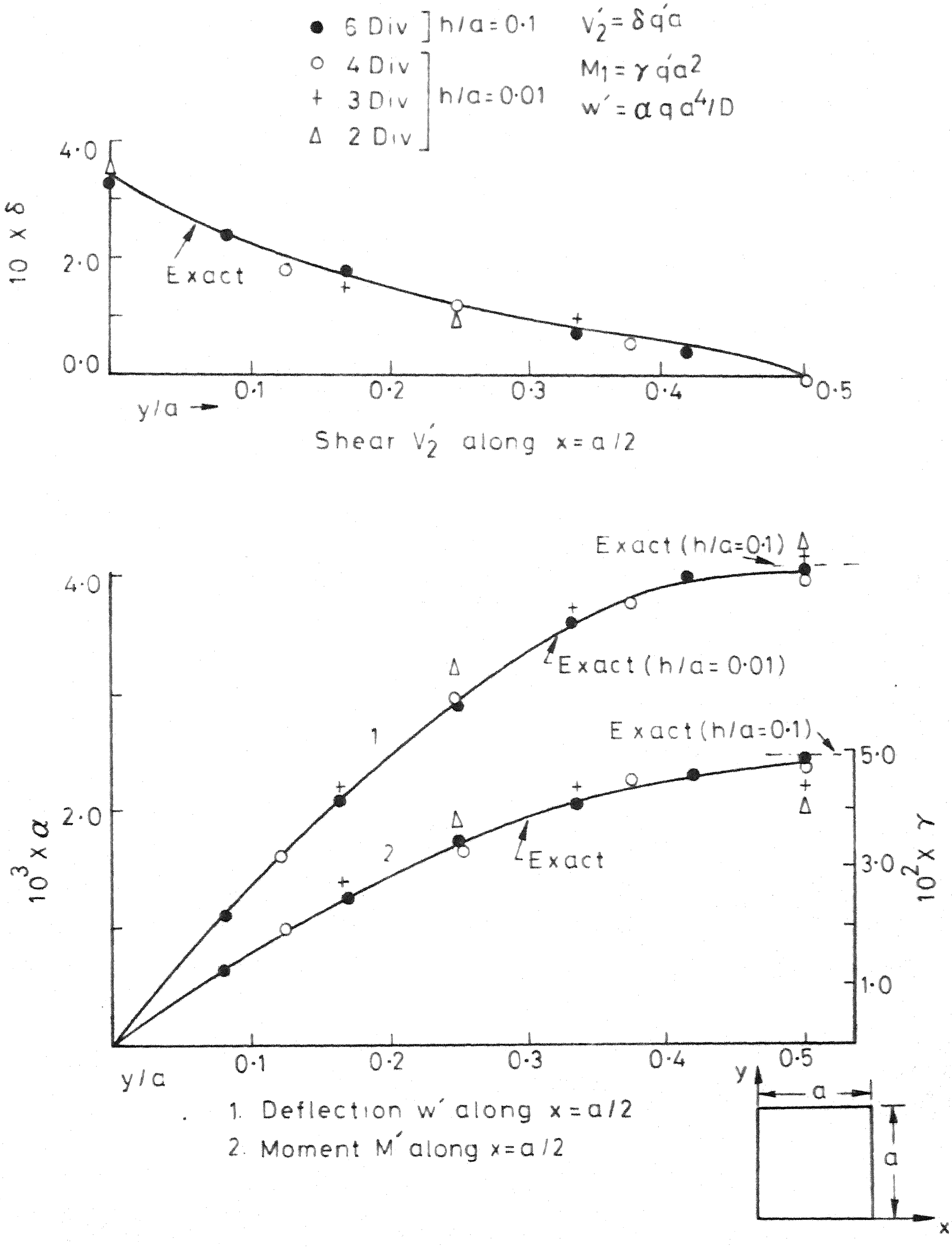


FIG. (6-5) DEFLECTION, MOMENT AND SHEAR OF S.S. SQ. PLATE

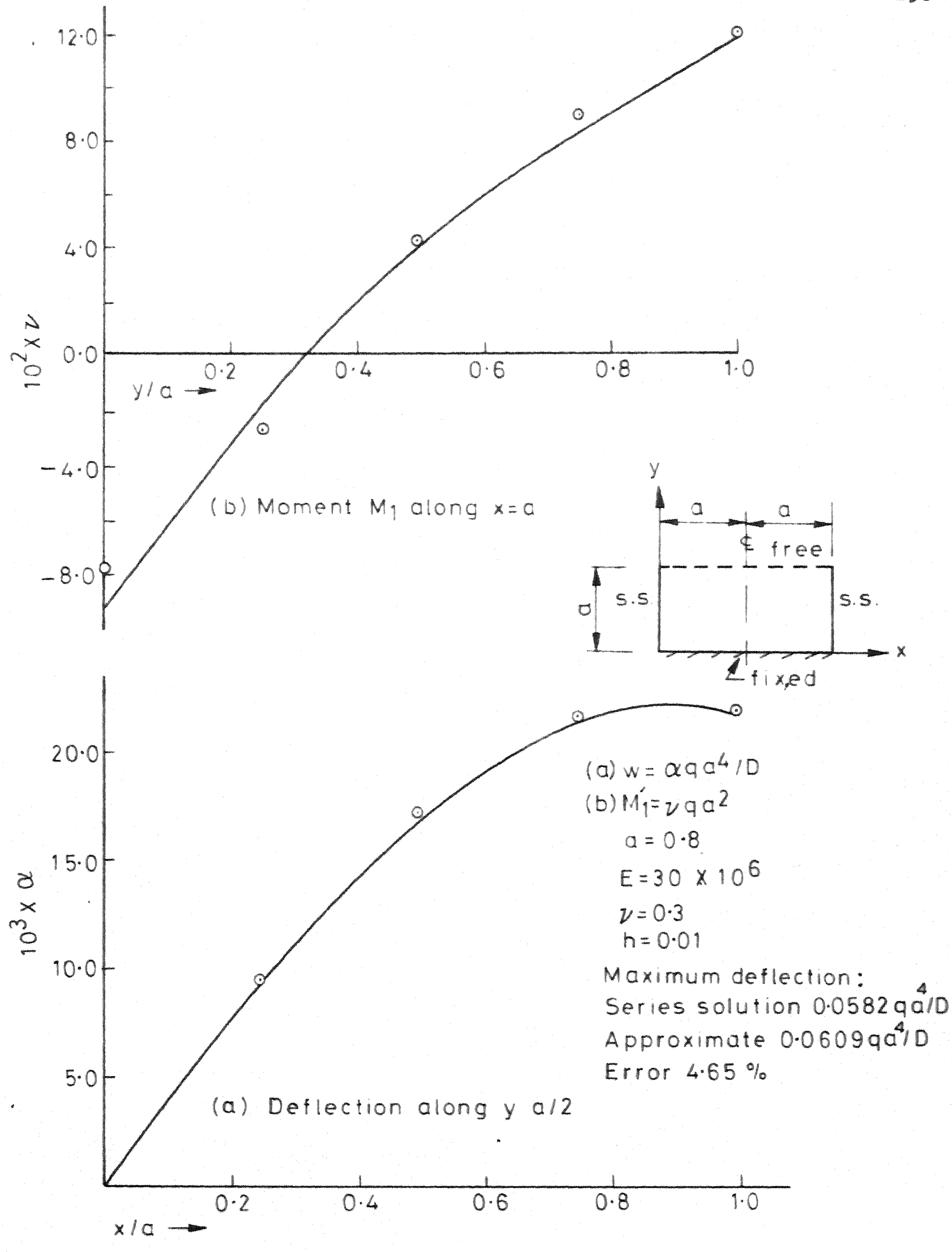


FIG. (6-6) DEFLECTION AND MOMENT FOR RECTANGULAR PLATE (THIN)

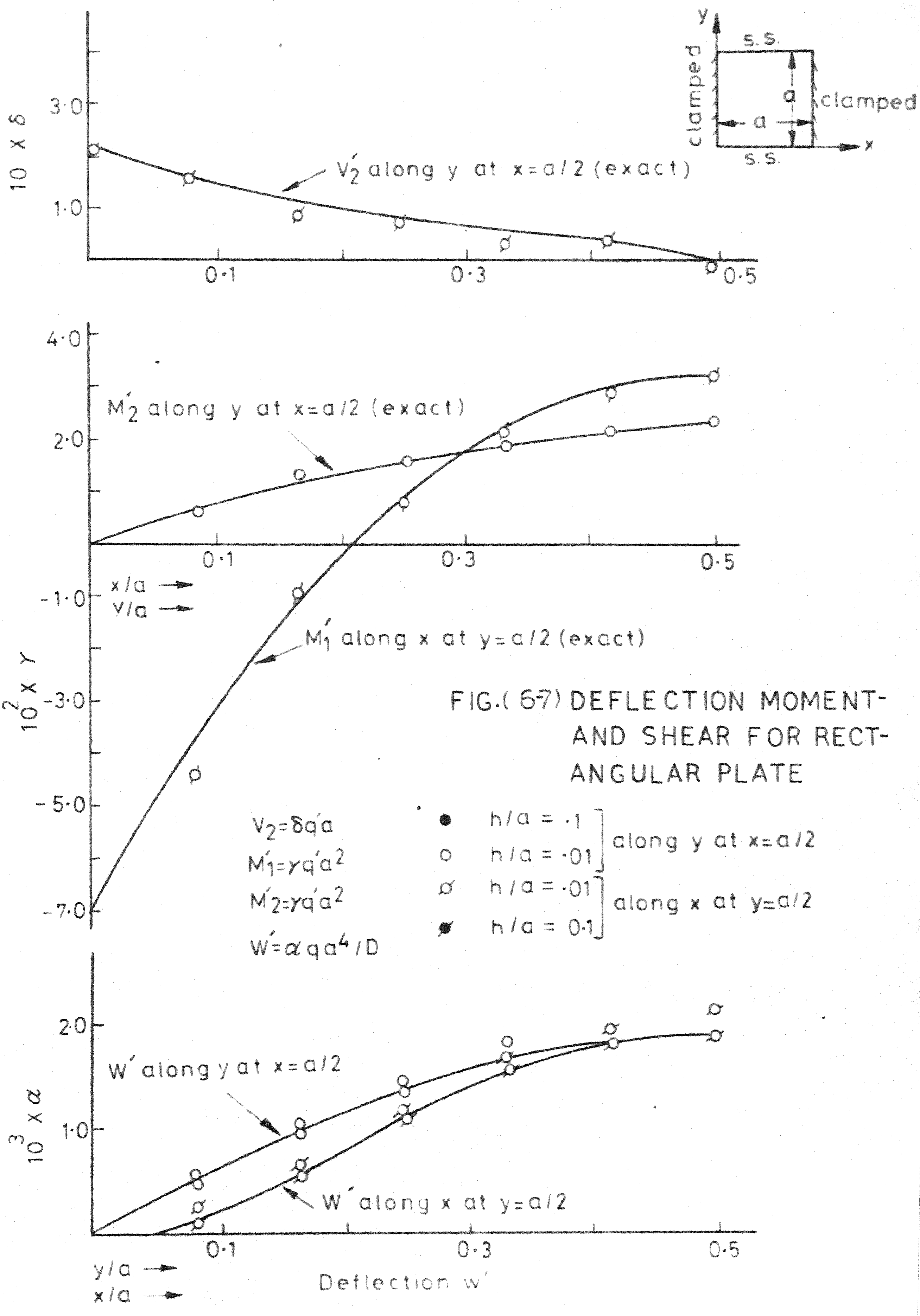


FIG.(6·7) DEFLECTION MOMENT- AND SHEAR FOR RECT-ANGULAR PLATE

Example	Coeff. for Max Deflection, α			Remarks
	Exact	Finite Ele. Solution	Finite Elec. solution with error correction	
1a	0.00406	0.00411	0.00380	(Fig.6.5) For thin s.s. plate with 4 x 4 Div.
1b	0.00424	0.00428	0.00417	(Fig.6.5) For s.s. plate with h/s = 0.1 and 6 x 6 Div.
2	0.0582	0.0609	0.0496	(Fig.6-6) For thin plate with 4 x 4 Div.
3	0.00192	0.00197	0.00183	(Fig.6-7) For thin plate with 6 x 6 Div.

Table 6-6 : Approximate error correction associated with maximum deflection, $w' = \alpha q a^4/D$.

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CHAPTER 7

SHELL ANALYSIS

7.1 Introduction:

In this Chapter, finite element equations for shell structures will be derived through Reissner's energy approach. This will be a continuation of the previous chapter.

Many investigators have already performed the stress analysis of shells using finite element technique with varying degree of success. Since, many references are available on this study, only a few recent investigations will be mentioned here. Clough and Johnson[1], Carr[2], Argyris et al.[3] and others have used flat , quadrilateral and triangular elements for the shell discretization. But for curved structures, curved elements are desirable because they eliminate or at least minimize the error involved in modelling the structure.

An almost completely compatible curved triangular element has been demonstrated by Bonnes, Dhatt, Giroux and Robichand[4]. For cylindrical shell, a completely compatible curved rectangular model has been developed by Bogner, Fox and Schmit[5]. Other notable contributors are Oden[6], Cantin and Clough[7]. Nonlinear analysis of shell structures have been done by Stricklin, Hsu and Pian[8], Schmit, Bogner and Fox[9], Wempner[10] and Saeed Yaghmai[11].

A mixed finite element model for linear[12] as well as for nonlinear[13] theory of shell have been developed by Prato via Reissner's principle. He has assumed linear displacements and moments over a triangular element. In this Chapter, the shell equations in general curvilinear coordinates will be discretized taking linear distribution of all the fifteen unknowns as obtained through Reissner's shell theory.

7.2 Governing Equation:

Consider a curvilinear coordinate system where (x, y) represent axes in the plane of the shell and z normal to the plane. Following Naghdi[14], the governing variational equation for the shell is given by,

$$\iint \{ -\delta U_x \left| \alpha_y \frac{\partial N_x}{\partial x} + N_x \frac{\partial \alpha_y}{\partial x} + \alpha_x \frac{\partial N_{yx}}{\partial y} + N_{yx} \frac{\partial \alpha_x}{\partial y} + N_{xy} \frac{\partial \alpha_x}{\partial y} \right.$$

$$\left. - N_y \frac{\partial \alpha_y}{\partial x} + \alpha_x \alpha_y \left(\frac{V_x}{R_x} + p_x \right) \right|$$

$$-\delta U_y \left| \alpha_y \frac{\partial N_{xy}}{\partial x} + N_{xy} \frac{\partial \alpha_y}{\partial x} + \alpha_x \frac{\partial N_y}{\partial y} + N_y \frac{\partial \alpha_x}{\partial y} + N_{yx} \frac{\partial \alpha_y}{\partial x} \right.$$

$$\left. - N_x \frac{\partial \alpha_x}{\partial y} + \alpha_x \alpha_y \left(\frac{V_y}{R_y} + p_y \right) \right|$$

$$-\delta W \left| \alpha_y \frac{\partial V_x}{\partial x} + V_x \frac{\partial \alpha_y}{\partial x} + \alpha_x \frac{\partial V_y}{\partial y} + V_y \frac{\partial \alpha_x}{\partial y} - \alpha_x \alpha_y \left(\frac{N_x}{R_x} + \frac{N_y}{R_y} \right) \right.$$

$$\left. + \alpha_x \alpha_y (q^+ H^+ - q^- H^-) \right|$$

$$-\delta \beta_x \left| \alpha_y \frac{\partial M_x}{\partial x} + M_x \frac{\partial \alpha_y}{\partial x} + \alpha_x \frac{\partial M_{yx}}{\partial y} + M_{yx} \frac{\partial \alpha_x}{\partial y} + M_{xy} \frac{\partial \alpha_x}{\partial y} \right.$$

$$\left. - M_y \frac{\partial \alpha_y}{\partial x} - \alpha_x \alpha_y (V_x - m_x) \right|$$

$$-\delta\beta_y \mid \alpha_y \frac{\partial M_{xy}}{\partial x} + M_{xy}\frac{\partial \alpha_y}{\partial x} + \alpha_x \frac{\partial M_y}{\partial y} + M_y \frac{\partial \alpha_x}{\partial y} + M_{yx}\frac{\partial \alpha_y}{\partial x}$$

$$- M_x \frac{\partial \alpha_x}{\partial y} - \alpha_x\,\alpha_y\,(V_y - m_y) \mid$$

$$+\delta N_x \mid \frac{1}{\alpha_x}\left(\frac{\partial U_x}{\partial x} + \frac{U_y}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) + \frac{W}{R_x} - \frac{1}{Eh}(N_x\,CF_2 - \nu N_y + M_x\,CF_1) \mid$$

$$+\delta N_y \mid \frac{1}{\alpha_y}\left(\frac{\partial U_y}{\partial y} + \frac{U_x}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) + \frac{W}{R_y} - \frac{1}{Eh}(N_y\,CF_3 - \nu N_x - M_y\,CF_1) \mid$$

$$+\delta N_{xy} \mid \frac{1}{\alpha_x}\left(\frac{\partial U_y}{\partial x} - \frac{U_x}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) + \frac{1}{\alpha_y}\left(\frac{\partial U_x}{\partial y} - \frac{U_y}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) CF_2 + \frac{1}{\alpha_y}\left(\frac{\partial \beta_x}{\partial y}\right.$$

$$\left. - \frac{\beta_y}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right)\frac{h^2}{12}\,CF_1 - \frac{2(1+\nu)}{Eh}(N_{xy}\,CF_2 + M_{xy}\,CF_1) \mid$$

$$+\delta N_{yx} \mid \frac{1}{\alpha_y}\left(\frac{\partial U_x}{\partial y} - \frac{U_y}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) + \frac{1}{\alpha_x}\left(\frac{\partial U_y}{\partial x} - \frac{U_x}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) CF_3 - \frac{1}{\alpha_x}\left(\frac{\partial \beta_y}{\partial x}\right.$$

$$\left. - \frac{\beta_x}{\alpha_y}\frac{\partial \alpha_x}{\partial\ y}\right)\frac{h^2}{12}\,CF_1 - \frac{2(1+\nu)}{Eh}(N_{yx}\,CF_3 - M_{yx}\,CF_1)$$

$$+\delta M_x \mid \frac{1}{\alpha_x}\left(\frac{\partial \beta_x}{\partial x} + \frac{\beta_y}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) - \frac{1}{Eh}\left(\frac{M_x}{h^2/12}\,CF_4 - \nu\frac{M_y}{h^2/12} + N_x\,CF_1\right)$$

$$+\delta M_y \mid \frac{1}{\alpha_y}\left(\frac{\partial \beta_y}{\partial y} + \frac{\beta_x}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) - \frac{1}{Eh}\left(\frac{M_y}{h^2/12}\,CF_5 - \nu\frac{M_x}{h^2/12} - N_y\,CF_1\right)$$

$$+\delta M_{xy} \mid \frac{1}{\alpha_x}\left(\frac{\partial \beta_y}{\partial x} - \frac{\beta_x}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) + \frac{1}{\alpha_y}\left(\frac{\partial \beta_x}{\partial y} - \frac{\beta_y}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) CF_4 + \frac{1}{\alpha_y}\left(\frac{\partial U_x}{\partial y}\right.$$

$$\left. - \frac{U_y}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) CF_1 - \frac{2(1+\nu)}{Eh}\left(\frac{M_{xy}}{h^2/12}\,CF_4 + N_{xy}\,CF_1\right)$$

$$+\delta M_{yx} \mid \frac{1}{\alpha_y}\left(\frac{\partial \beta_x}{\partial y} - \frac{\beta_y}{\alpha_x}\frac{\partial \alpha_y}{\partial x}\right) + \frac{1}{\alpha_x}\left(\frac{\partial \beta_y}{\partial x} - \frac{\beta_x}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) CF_5 - \frac{1}{\alpha_x}\left(\frac{\partial U_y}{\partial x}\right.$$

$$\left. - \frac{U_x}{\alpha_y}\frac{\partial \alpha_x}{\partial y}\right) CF_1 - \frac{2(1+\nu)}{Eh}\left(\frac{M_{yx}}{h^2/12}\,CF_5 - N_{yx}\,CF_1\right)$$

$$+\delta V_x \Big| \frac{1}{\alpha_x}\frac{\partial W}{\partial x} - \frac{U_x}{R_x} + \beta_x - 4.5\,\frac{1+\nu}{Eh}\,V_x\,CF_6 + \frac{1+\nu}{5E}\,\Big\{\frac{m_x}{h/2}\,\Big(1 + \frac{h^2}{28R_y}\,CF_1\Big) - \frac{h}{2}\,p_x\,CF_1\Big\}\Big|$$

$$+\delta V_y \Big| \frac{1}{\alpha_y}\frac{\partial W}{\partial y} - \frac{U_y}{R_y} + \beta_y - 4.5\,\frac{1+\nu}{Eh}\,V_y\,CF_7 + \frac{1+\nu}{5E}\,\Big\{\frac{m_y}{h/2}\,\Big(1 - \frac{h^2}{28R_x}\,CF_1\Big) + \frac{h}{2}\,p_y\,CF_1\Big\}\Big|\Big\}\,dx\,dy \tag{7.2-1}$$

In Eq. (7.2-1), E is Young's modulus, ν is Poission's ratio and h is the thickness of shell; αx and αy are Lame's parameters for curvilinear surface and R_x and R_y are principal radii of curvatures of the middle surface; U_x, U_y, β_x, β_y and W are displacement components, whereas N_x, N_y, N_{xy}, N_{yx}, V_x, V_y, M_x, M_y, M_{xy} and M_{yx} are stress resultants; p_x and p_y are membrane loads; q^+ and q^- are normal loads on the positive and negative sides of the shell element with respect to the thickness of the element. Over and above, the following notations have been used in Eq. (7.2-1):

$$CF_1 = \frac{1}{R_x} - \frac{1}{R_y}\,; \qquad CF_2 = 1-h^2\,CF_1/12\,R_y$$

$$CF_3 = 1 + h^2\,CF_1/12\,R_x \quad CF_4 = 1-3h^2\,CF_1/20\,R_y$$

$$CF_3 = 1 + 3h^2\,CF_1/20R_x \quad CF_6 = 8/15 - 2h^2\,CF_1/105\,R_y$$

$$CF_7 = 8/15 + 2h^2\,CF_1/105\,R_x$$

$$\text{and, } H^+ = 1+h(1/R_x + 1/R_y)/2 + h^2/4R_x\,R_y$$

$$H^- = 1-h(1/R_x + 1/R_y)/2 + h^2/4R_x\,R_y \tag{7.2-2}$$

For more elaborate derivation of the Eq. (7.2-1), see reference 14.

7.3 Finite Element Derivation:

Assume linear distribution (in the curvilinear coordinates x, y) over a curved triangular element for all the fifteen field variables. Considering a typical quantity H and its nodal values as H_1, H_2 and H_3, the interpolating function for H will be given by,

$$H = \overset{*}{T}_1 H_1 + \overset{*}{T}_2 H_2 + \overset{*}{T}_3 H_3 \qquad (7.3\text{-}1)$$

where,

$$\begin{aligned} \overset{*}{T}_1 &= T_{11} + T_{21}\, x + T_{31}\, y \\ \overset{*}{T}_2 &= T_{12} + T_{22}\, x + T_{32}\, y \\ \overset{*}{T}_3 &= T_{13} + T_{23}\, x + T_{33}\, y \end{aligned} \qquad (7.3\text{-}2)$$

$$|T| = \frac{1}{2A} \begin{bmatrix} \frac{2}{3}A & \frac{2}{3}A & \frac{2}{3}A \\ a_{22}-a_{32} & a_{32} & -a_{22} \\ a_{31}-a_{21} & -a_{31} & a_{21} \end{bmatrix} \qquad (7.3\text{-}3)$$

$$2A = a_{21}\, a_{32} - a_{31}\, a_{22}$$

and a_{ij}'s are curvilinear lengths, similar to those defined in Eq. (6.3-4). Substituting the distribution (7.3-1) in Eq. (7.2-1), the following discretized form of the equations are obtained:

$$\Sigma' \left| N_{xk}(T_{2k}\, I_2^i + I_{20}^{ik}) + N_{yxk}(T_{3k}\, I_1^i + I_{19}^{ik}) + N_{xyk}\, I_{19}^{ik} \right.$$

$$\left. - N_{yk}\, I_{20}^{ik} + V_{xk}\, I_{17}^{ik} + LI_1^i \right| = 0 \qquad (7.3\text{-}4)$$

$$\Sigma' \left| N_{xyk}\, (T_{2k}\, I_2^i + I_{20}^{ik}) + N_{yk}\, (T_{3k}\, I_1^i + I_{19}^{ik}) + N_{yxk}\, I_{20}^{ik} \right.$$

$$\left. - N_{xk}\, I_{19}^{ik} + V_{yk}\, I_{18}^{ik} + LI_2^i \right| = 0 \qquad (7.3\text{-}5)$$

$$\Sigma' \left| V_{xk}(T_{2k}\, I_2^i + I_{20}^{ik}) + V_{yk}(T_{3k}\, I_1^i + I_{19}^{ik}) - N_{xk}\, I_{17}^{ik} \right.$$

$$\left. - N_{yk}\, I_{18}^{ik} + LI_3^i \right| = 0 \qquad (7.3\text{-}6)$$

$$\Sigma' \left| M_{xk}\, (T_{2k}\, I_2^i + I_{20}^{ik}) + M_{yxk}\, (T_{3k}\, I_1^i + I_{19}^{ik}) + M_{xyk}\, I_{19}^{ik} \right.$$

$$\left. - M_{yk}\, I_{20}^{ik} - V_{xk}\, I_{14}^{ik} + LI_4^i \right| = 0 \qquad (7.3\text{-}7)$$

$$\Sigma' \left| M_{xyk}\, (T_{2k}\, I_2^i + I_{20}^{ik}) + M_{yk}\, (T_{3k}\, I_1^i + I_{19}^{ik}) + M_{yxk}\, I_{20}^{ik} \right.$$

$$\left. - M_{xk}\, I_{19}^{ik} - V_{yk}\, I_{14}^{ik} + LI_5^i \right| = 0 \qquad (7.3\text{-}8)$$

$$\Sigma' \left| U_{xk}\, (T_{2k}\, I_3^i) + U_{yk}\, I_{23}^{ik} + W_k\, I_{15}^{ik} - N_{xk}\, I_{33}^{ik} + N_{yk}\, I_{25}^{ik} \right.$$

$$\left. - M_{xk}\, I_{27}^{ik} \right| = 0 \qquad (7.3\text{-}9)$$

$$\Sigma' \left| U_{yk}\, (T_{3k}\, I_4^i) + U_{xk}\, I_{24}^{ik} + W_k\, I_{16}^{ik} - N_{yk}\, I_{36}^{ik} + N_{xk}\, I_{25}^{ik} \right.$$

$$\left. + M_{yk}\, I_{27}^{ik} \right| = 0 \qquad (7.3\text{-}10)$$

$$\Sigma' \left| U_{xk}\, (T_{3k}\, I_{10}^i - I_{23}^{ik}) + U_{yk}\, (T_{2k}\, I_3^i - I_{35}^{ik}) + \beta_{xk}\, T_{3k}\, I_8^i \right.$$

$$\left. - \beta_{yk}\, I_{32}^{ik} - N_{xyk}\, I_{34}^{ik} - M_{xyk}\, I_{28}^{ik} \right| = 0 \qquad (7.3\text{-}11)$$

$$\Sigma' \left| U_{xk}\, (T_{3k}\, I_4^i - I_{38}^{ik}) + U_{yk}\, (T_{2k}\, I_9^i - I_{24}^{ik}) + \beta_{xk}\, I_{31}^{ik} \right.$$

$$\left. - \beta_{yk}\, T_{2k}\, I_7^i - N_{yxk}\, I_{37}^{ik} + M_{yxk}\, I_{28}^{ik} \right| = 0 \qquad (7.3\text{-}12)$$

$$\Sigma' \left| \beta_{xk} T_{2k} I_3^i + \beta_{yk} I_{23}^{ik} - M_{xk} I_{39}^{ik} + M_{yk} I_{26}^{ik} - N_{xk} I_{27}^{ik} \right| = 0 \tag{7.3-13}$$

$$\Sigma' \left| \beta_{xk} I_{24}^{ik} + \beta_{yk} T_{3k} I_4^i + M_{xk} I_{26}^{ik} - M_{yk} I_{42}^{ik} + N_{yk} I_{27}^{ik} \right| = 0 \tag{7.3-14}$$

$$\Sigma' \left| \beta_{xk} (T_{3k} I_{12}^i - I_{23}^{ik}) + \beta_{yk} (T_{2k} I_3^i - I_{41}^{ik}) + U_{xk} T_{3k} I_6^i - U_{yk} I_{30}^{ik} - M_{xyk} I_{40}^{ik} - N_{xyk} I_{28}^{ik} \right| = 0 \tag{7.3-15}$$

$$\Sigma' \left| \beta_{xk} (T_{3k} I_4^i - I_{44}^{ik}) + \beta_{yk} (T_{2k} I_{11}^i - I_{24}^{ik}) + U_{xk} I_{29}^{ik} - U_{yk} T_{2k} I_5^i - M_{yxk} I_{43}^{ik} + N_{yxk} I_{28}^{ik} \right| = 0 \tag{7.3-16}$$

$$\Sigma' \left| W_k T_{2k} I_3^i - U_{xk} I_{15}^{ik} + \beta_{xk} I_{13}^{ik} - V_{xk} I_{45}^{ik} + LI_6^i \right| = 0 \tag{7.3-17}$$

$$\Sigma' \left| W_k T_{34} I_4^i - U_{yk} I_{16}^{ik} + \beta_{yk} I_{13}^{ik} - V_{yk} I_{46}^{ik} + LI_7^i \right| = 0 \tag{7.3-18}$$

In the Eqs. (7.3-4) to (7.3-18), the summation of the repeated index k has been assumed and k takes the values of 1, 2 and 3. U_{xk}, U_{yk}, β_{xk}, β_{yk} and W_k represent the displacements at the k-th node, whereas nodal stresses are denoted as N_{xk}, N_{yk}, N_{xyk}, N_{yxk}, M_{xk}, M_{yk}, M_{xyk}, M_{yxk}, V_{xk} and Y_{yk}. The loading integrals LI_1^i to LI_7^i and other integrations, I_1^i to I_{12}^i and I_{13}^{ij} to I_{46}^{ij} have been defined in Table 7-1, where i, j vary from 1 to 3. Also, Σ' in Eqs. (7.3-4) to (7.3-18) denotes the summation over all the elements around a particular node i. These equations can be solved in the similar way, indicated in Chapter 6.

7.4 Note for Numerical Example:

A computer program has been developed for the shell analysis which has fifteen degrees of freedom per node. Due to limitation in memory space, the program can accommodate only a half band width of four node points. For this severe restriction, it has not been possible to solve any problem of practical interest. However, for few special geometries, the resulting matrices have been checked from statical consideration.

Table 7-1 Integrals

$$LI_1^i = \iint \overset{*}{T}_i \ \alpha_x \ \alpha_y \ p_x \ dx \ dy$$

$$LI_2^i = \iint \overset{*}{T}_i \ \alpha_x \ \alpha_y \ p_y \ dx \ dy$$

$$LI_3^i = \iint \overset{*}{T}_i \ \alpha_x \ \alpha_y \ (q^+H^+ - q^-H^-) \ dx \ dy$$

$$LI_4^i = \iint \overset{*}{T}_i \ \alpha_x \ \alpha_y \ m_x \ dx \ dy$$

$$LI_5^i = \iint \overset{*}{T}_i \ \alpha_x \ \alpha_y \ m_y \ dx \ dy$$

$$LI_6^i = \iint \left| \ m_x \ (1+h^2 \ CF_1/28R_y)/ \ \frac{h}{2} - p_x \ h \ CF_1/2 \ \right| \ (1+\nu)/5E \ \ dx \ dy$$

$$LI_7^i = \iint \left| \ m_y \ (1-h^2 \ CF_1/28R_x)/ \ \frac{h}{2} + p_y \ h \ CF_1/2 \ \right| \ (1+\nu)/5E \ \ dx \ dy$$

$$I_1^i = \iint \overset{*}{T}_i \ \alpha_x \ dx \ dy$$

$$I_2^i = \iint \overset{*}{T}_i \ \alpha_y \ dx \ dy$$

$$I_3^i = \iint \overset{*}{T}_i/\alpha_x \ dx \ dy$$

$$I_4^i = \iint \overset{*}{T}_i/\alpha_y \ dx \ dy$$

$$I_5^i = \iint \overset{*}{T}_i \ CF_1/\alpha_x \ dx \ dy$$

$$I_6^i = \iint \overset{*}{T}_i \ CF_1/\alpha_y \ dx \ dy$$

$$I_7^i = \iint \overset{*}{T}_i \ h^2 \ CF_1/12 \ \alpha_x \ dx \ dy$$

$$I_8^i = \iint \overset{*}{T}_i \ h^2 \ CF_1/12 \ \alpha_y \ dx \ dy$$

$$I_9^i = \iint \overset{*}{T}_i \ CF_3/\alpha_x \ \ dx \ dy$$

$$I_{10}^i = \iint \overset{*}{T}_i \ CF_2/\alpha_y \ \ dx \ dy$$

$$I_{11}^{i} = \iint \overset{*}{T}_i \; CF_5/\alpha_x \; dx\, dy$$

$$I_{12}^{i} = \iint \overset{*}{T}_i \; CF_4/\alpha_y \; dx\, dy$$

$$I_{13}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; dx\, dy$$

$$I_{14}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; \alpha_x \; \alpha_y \; dx\, dy$$

$$I_{15}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j/R_x \; dx\, dy$$

$$I_{16}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j/R_y \; dx\, dy$$

$$I_{17}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; \alpha_x \; \alpha_y/R_x \; dx\, dy$$

$$I_{18}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; \alpha_x \; \alpha_y/R_y \; dx\, dy$$

$$I_{19}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; \partial\alpha_x/\partial y \; dx\, dy$$

$$I_{20}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; \partial\alpha_y/\partial x \; dx\, dy$$

$$I_{21}^{ij} = \iint (\overset{*}{T}_i \; \overset{*}{T}_j/\alpha_y) \; \partial\alpha_x/\partial y \; dx\, dy$$

$$I_{22}^{ij} = \iint (\overset{*}{T}_i \; \overset{*}{T}_j/\alpha_x) \; \partial\alpha_y/\partial x \; dx\, dy$$

$$I_{23}^{ij} = \iint (\overset{*}{T}_i \; \overset{*}{T}_j/\alpha_x \; \alpha_y) \; \partial\alpha_x/\partial y \; dx\, dy$$

$$I_{24}^{ij} = \iint (\overset{*}{T}_i \; \overset{*}{T}_j/\alpha_x \; \alpha_y) \; \partial\alpha_y/\partial x \; dx\, dy$$

$$I_{25}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; \nu/Eh \; dx\, dy$$

$$I_{26}^{ij} = \iint 12 \; \overset{*}{T}_i \; \overset{*}{T}_j \; \nu/Eh^3 \; dx\, dy$$

$$I_{27}^{ij} = \iint \overset{*}{T}_i \; \overset{*}{T}_j \; CF_1/Eh \; dx\, dy$$

$$I_{28}^{ij} = \iint 2 \; \overset{*}{T}_i \; \overset{*}{T}_j \; CF_1 \; (1+\nu)/Eh \; dx\, dy$$

$$I_{29}^{ij} = \iint (\overset{*}{T}_i \; \overset{*}{T}_j \; CF_1/\alpha_x \; \alpha_y) \; \partial\alpha_x/\partial y \; dx\, dy$$

$$I_{30}^{ij} = \iint (\overset{*}{T}_i \; \overset{*}{T}_j \; CF_1/\alpha_x \; \alpha_y) \; \partial\alpha_y/\partial x \; dx\, dy$$

$$I_{31}^{ij} = \iint (\overset{*}{T}_i \overset{*}{T}_j CF_1 h^2/12 \alpha_x \alpha_y) \partial\alpha_x/\partial y \, dx \, dy$$

$$I_{32}^{ij} = \iint (\overset{*}{T}_i \overset{*}{T}_j CF_1 h^2/12 \alpha_x \alpha_y) \partial\alpha_y/\partial x \, dx \, dy$$

$$I_{33}^{ij} = \iint \overset{*}{T}_i \overset{*}{T}_j CF_2/Eh \, dx \, dy$$

$$I_{34}^{ij} = \iint 2 \overset{*}{T}_i \overset{*}{T}_j CF_2 (1+\nu)/Eh \, dx \, dy$$

$$I_{35}^{ij} = \iint (\overset{*}{T}_i \overset{*}{T}_j CF_2/\alpha_x \alpha_y) \partial\alpha_y/\partial x \, dx \, dy$$

$$I_{36}^{ij} = \iint \overset{*}{T}_i \overset{*}{T}_j CF_3/Eh \, dx \, dy$$

$$I_{37}^{ij} = \iint 2 \overset{*}{T}_i \overset{*}{T}_j CF_3 (1+\nu)/Eh \, dx \, dy$$

$$I_{38}^{ij} = \iint (\overset{*}{T}_i \overset{*}{T}_j CF_3/\alpha_x \alpha_y) \partial\alpha_x/\partial y \, dx \, dy$$

$$I_{39}^{ij} = \iint 12 \overset{*}{T}_i \overset{*}{T}_j CF_4/Eh^3 \, dx \, dy$$

$$I_{40}^{ij} = \iint 24 \overset{*}{T}_i \overset{*}{T}_j CF_4/(1+\nu)/Eh^3 \, dx \, dy$$

$$I_{41}^{ij} = \iint (\overset{*}{T}_i \overset{*}{T}_j CF_4/\alpha_x \alpha_y) \partial\alpha_y/\partial x \, dx \, dy$$

$$I_{42}^{ij} = \iint 12 \overset{*}{T}_i \overset{*}{T}_j CF_5/Eh^3 \, dx \, dy$$

$$I_{43}^{ij} = \iint 24 \overset{*}{T}_i \overset{*}{T}_j CF_5(1+\nu)/Eh^3 \, dx \, dy$$

$$I_{44}^{ij} = \iint (\overset{*}{T}_i \overset{*}{T}_j CF_5/\alpha_x \alpha_y) \partial\alpha_x/\partial y \, dx \, dy$$

$$I_{45}^{ij} = \iint 4.5 (1+\nu) \overset{*}{T}_i \overset{*}{T}_j CF_6/Eh \, dx \, dy$$

$$I_{46}^{ij} = \iint 4.5 (1+\nu) \overset{*}{T}_i \overset{*}{T}_j CF_7/Eh \, dx \, dy$$

7.5 References:

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CHAPTER 8

CONCLUSION AND COMMENTS ON FUTURE RESEARCH

8.1 Conclusion:

In this sequel, following the concept forwarded by Green and Rivlin (Ref. 4 of Chapter 1), the general governing equations for a continuum have been derived on the basis of thermodynamic laws and material frame indifference principle. These equations can be directly applied for finite element discretization. By suitable manipulation, displacement, force or mixed model can be obtained separately. The equations essentially represent Galerkin-type approximation applied on Cauchy's laws of motion and on equation of thermal state. Application of finite element method discretizing the space variables, yield, in general, a set of nonlinear ordinary differential equations in terms of the unknown field variables. When parametric differentiation technique is applied, the resulting modified equations become linear ordinary differential equations of variable coefficients in terms of the differentials of the field variables with respect to a parameter and can be conveniently integrated by any standard numerical technique. The unknown field variables can now be obtained by employing any quadrature formula. In essence, the nonlinearity of the problem is restricted only in the quadrature manipulation which virtually poses no difficulty.

The scope of application of these equations is very wide. Actually, with minor modifications, they can be applied to any particular cases of continuum problems, for example, multipolar cases, viscoelasticity, viscoplasticity, coupled thermoelasticity with dissipative properties etc. It appears that the applications to problems of nonlinear stability or complicated fluid flow with thermal effects may also be possible. Although, in this study stresses have been expressed as the functionals of strains and their time rates, the constitutive equations expressed in the other way, i.e., expressing the strains and their rates as the functionals of stresses and their time rates are also feasible. This will be rather straight forward for mixed model, but for other cases it may be quite complicated. The geometrical transformations, for problems like plates and shells can also be done in the usual manner.

In this study, various particular cases of simpler types have been solved utilising this general technique. The results, in general, show excellent agreement for one step solutions, i.e., for linear cases. The results are also quite satisfactory for short range processes. But for long range solutions, the incremental step length becomes a critical governing factor. Unfortunately, due to the restriction in available computer time, no further refinement in this direction has been possible. But it appears that it will

offer no more complexities than the numerical solution of nonlinear differential equations of initial value problems.

Lastly, it is felt necessary to comment on the variational techniques commonly applied for the derivation of finite element methods. As such in the present set up, the variational techniques seem to be a more favourite, possible because, the discontinuities along the interelement boundaries can be easily accounted and convergency can be proved on the basis of some potential. But, since the variational technique is fundamentally an heuristic approach, the modifications which can be incorporated there, may also be applied on the set of equations derived in this variant. Though, through the concept of potential it is convenient to prove convergence in the mean, this is not always necessary. Moreover, for the proof, a requisite predefined norm can also be employed. For example, in Chapter 6 the convergency of the plate bending model has been shown by employing the classical 'order of accuracy' technique, although its scope of applicability is rather limited. However, it necessitates more thorough and extensive investigations.

8.2 Comments on Future Research:

There is so much work yet to be done on continuum mechanics problems, that it is difficult to particularise any one of them. Except a few exceptions, the multipolar cases

have remained practically untouched. Moreover, the time dependent and nonlinear effects on plates and shells are also getting more and more prominance for more accurate analysis. Actually, all the problems dealt here also require more intensive and thorough studies.

APPENDIX A

A.1 A Note on the Computer Programs:

Separate computer programs have been developed for the types of problems dealt in Chapters 3 to 7. Their underlying common feature will be described here.

Each program can be subdivided into the sections: (1) Control, (2) Stiffness, (3) Assembly, (4) Elimination (5) Input/Output and (6) Auxiliaries. The execution will start through control. It first calls the I/O section to supply initial informations. Then it sets the necessary indices for the nodes as well as elements, for example it generates the number of elements surrounding a node. At the same time, the control calls the stiffness which supply the matrices through a logical tape unit. When this is over, the control calls the Assembly where the matrices which were supplied elementwise will be sorted nodewise. Then the associated matrices are assembled for each node and immediately, the Elimination section is called to eliminate the matrices upto the diagonal term making it an identity matrix. The left over portion, i.e., half band width behind the diagonal, will be preserved on a logical tape unit for the back sweep to evaluate the nodal unknowns. The simple Gauss elimination procedure has been employed here. During this execution logical tape units u00, u01 of IBM 7044 computer are utilised to save memory

space. Once the nodal unknowns are known, various other auxiliary factors can be calculated through subroutines which have been grouped as Auxiliaries.

This procedure may be associated with an integration technique coupled with a quadrature for solving the incremental processes. In this study, modified Euler procedure with trapezoidal rule has been incorporated.

The main disadvantage of this program is the excessive use of tapes which have made it less efficient on the basis of execution time.

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